

SPLV Lecture 4

①

- STLC & CCCs:

every c.c. pres., $\text{Down}(P) \cong \text{Pos}(P^{\text{op}}, \mathcal{P})$

$$\text{Psh}(e) \cong [e^{\text{op}}, \text{Set}]$$

$$\text{V-Psh}(e) \cong [e^{\text{op}}, \text{V}]$$

$$(1) \quad \underline{\alpha} : P \rightarrow \text{Down}(P)$$
$$p \mapsto \downarrow(p)$$

$$\underline{\alpha} : e \rightarrow [e^{\text{op}}, \text{Set}]$$
$$x \mapsto e(-, x)$$

$$S \in \text{Down}(P)$$

$$\cong \bigcup_{s \in S} \downarrow(s)$$

$$P \in \text{Psh}(e)$$

$$\cong \int_{\downarrow} \alpha(c)$$

$$(2) \quad \text{Sh}(e, \gamma) \subseteq \text{Psh}(e)$$

f.f.

(3) V-Cat

$$V = \Omega, \text{Pos},$$

$$V = \text{P}(L)$$

exercise

- linear logic models.

Rel $A \leftrightarrow B$
! \circlearrowright
Seely comonad

$$\begin{aligned} A \leftrightarrow B & \quad ! (A \times B) \multimap C \\ \otimes, \multimap & \quad \cong \\ \text{Seely iso} & \quad \cong \\ & \quad ! (A) \otimes ! (B) \multimap C \\ & \quad ! A \multimap (! B \multimap C) \end{aligned}$$

\circlearrowright (Coh, lin)
! (Coh, stable)

$$A \Rightarrow B \rightsquigarrow \frac{! A \multimap B}{\cdot}$$

CBN Girard translation.

* effects

$$\frac{\frac{\Gamma \vdash \text{print} : \mathbb{1}}{\Gamma \vdash f : \mathbb{1}}}{\Gamma \vdash f = * : \mathbb{1}}$$

print \neq * .

- ccc - a e, (1) $\mathcal{L}(\Gamma, 1) \cong 1$

(2) $\mathcal{L}(\Gamma, A \times B) \cong \mathcal{L}(\Gamma, A) \times \mathcal{L}(\Gamma, B)$

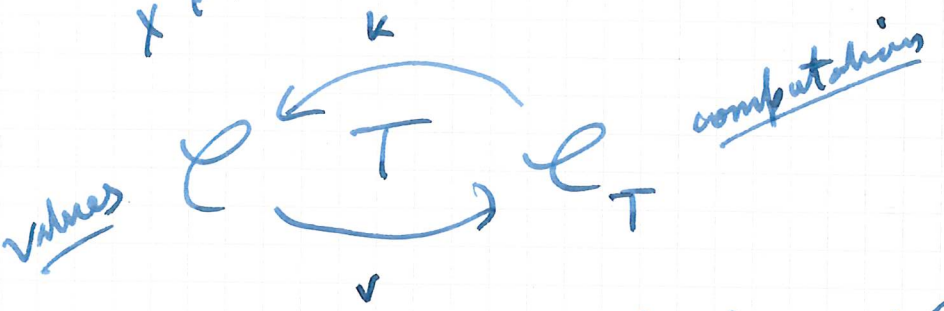
(3) $\mathcal{L}(\Gamma, A \Rightarrow B) \cong \mathcal{L}(\Gamma \times A, B)$.

- partial ccc, \mathcal{L}

$v: \mathcal{L} \rightarrow \mathcal{L}_T$

$k: \mathcal{L}_T \rightarrow \mathcal{L}$

$x \mapsto Tx$ strong monad, T



$v \dashv k$

$\mathcal{L}_T(A, B) = \mathcal{L}(A, TB)$

(1), (2) + Partial exponentials.
Kleisli

(3*) ~~XXXXXXXX~~

$\mathcal{L}_T(\Gamma, A \rightarrow TB) \cong \mathcal{L}_T(\Gamma \times A, TB)$
partial exp.

$\Delta \vdash G \xrightarrow{\Delta, T} \text{family } \underline{e}$
 $\Delta, T \vdash \text{interaction}$

$\Gamma \vdash p: \text{FilePtr} \quad \Gamma \vdash s: \text{Str}$

 $\Gamma \vdash \text{print}(p, s): \mathbb{I}$

- if $p: \text{FilePtr} \in \Gamma,$
 $\Gamma \vdash e: \mathbb{I},$

$\Gamma \supseteq \Delta, \quad \frac{\Delta \vdash e: A}{\Gamma \vdash e: A} \quad \text{if } p: \text{FilePtr} \notin \Gamma,$
 $\Gamma, t e = * : \mathbb{I}$

$p \rightsquigarrow$

Δ
 \vdots
 $\Gamma \vdash p: \text{Ptr}$

Γ

Fix set of names C ,

$$e^{cn} \text{ Obj} := (A : \text{Set}, H : \subseteq C \times A) \cong \underline{W_A : A \rightarrow \mathcal{P}(C)}$$

$$\text{Hom}((A, W_A), (B, W_B)) \quad W_A : A \rightarrow \mathcal{P}(C)$$

$$\left. \begin{array}{l} C \times A \rightarrow 2 \\ \cong \\ A \rightarrow 2^C \end{array} \right\}$$

$$= \{ f : A \rightarrow B \mid \forall a \in A, W_B(f(a)) \subseteq W_A(a) \}$$

Fam(Set^{op})
containers.

$$\text{Fam}(\mathcal{P}(C), \subseteq)$$

Kripke Sheaves.

$$(\mathcal{P}(C), \mathcal{J})$$

complete
Heyting
alg H.

$$\alpha : H \longrightarrow \underline{\text{Fam}(H)}$$

exercise.

- \mathcal{C} has products, exponentials, coproducts.

$$\begin{aligned}
 (*) \quad & (A, w_A) \times (B, w_B) \\
 & = (A \times B, w_{A \times B}(a, b) = w_A(a) \cup w_B(b))
 \end{aligned}$$

$$(*) \quad (A, w_A) \Rightarrow (B, w_B) \quad \text{(point wise)}$$

exercise

(*) coproducts (free).

- STLC + coproducts.
 $(0, 1, +, \times, \Rightarrow)$

$$- T : \mathcal{C}^{\text{ch}} \rightarrow \mathcal{C}^{\text{ch}} \quad (\text{strong monoidal})$$

$$(X, w_X) \mapsto (X \times (\text{Ch} \rightarrow \underline{\text{Str}}), w_{T(X)})$$

$$\text{Ch} \in \mathcal{C}^{\text{ch}}$$

$$\text{Ch}, w_{\text{Ch}} : \text{Ch} \rightarrow \mathcal{P}(\text{Ch})$$

$$x \mapsto \{x\}$$

$$w_{T(X)} : (X \times (\text{Ch} \rightarrow \underline{\text{Str}})) \rightarrow \mathcal{P}(\text{Ch})$$

$$(x, o) \mapsto w_X(x) \cup \{c \mid o(c) \neq \varepsilon\}$$

$$- \square : \mathcal{C}^{\text{ch}} \rightarrow \mathcal{C}^{\text{ch}} \quad (\text{strong monoidal comonoid})$$

$$(X, w_X) \mapsto (\{x \in X \mid w_X(x) = \emptyset\}, w_{\square X})$$

$$- \phi : \square T(A) \xrightarrow{\sim} \square A$$

$$\varepsilon_A : \square A \rightarrow A$$

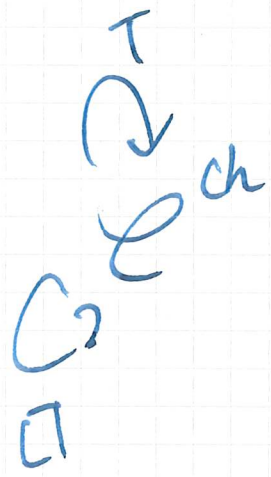
$$\delta_A : \square A \xrightarrow{\sim} \square^2 A$$

$$\square(A \times B)$$

$$\cong$$

$$\square A \times \square B$$

(7)



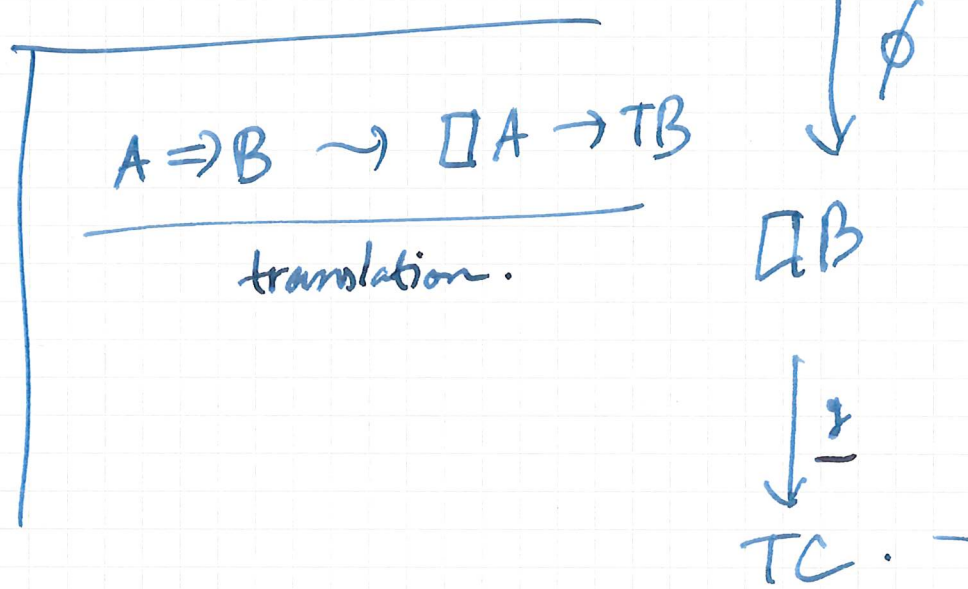
$$\mathcal{L}_T^{\Delta}(A, B) = \mathcal{L}^{\Sigma}(\Delta A, \Gamma B)$$

$$f: \Delta A \rightarrow \Gamma B$$

$$g: \Delta B \rightarrow \Gamma C$$

$$\begin{aligned} &\mathcal{L}_T^{\Delta}(A \times B, C) \\ \equiv &\mathcal{L}(\Delta(A \times B), \Gamma C) \\ \cong &\mathcal{L}^{\Sigma}(\Delta A \times \Delta B, \Gamma C) \\ \cong &\mathcal{L}^{ch}(\Delta A, \underline{\Delta B \rightarrow \Gamma C}) \end{aligned}$$

$$\begin{aligned} &\text{fig} \\ \equiv &\Delta A \xrightarrow{\sim} \Delta \Delta A \xrightarrow{\Delta f} \Delta \Gamma B \end{aligned}$$



$\Gamma, \Delta P$

$$\theta ::= \Delta \mid \langle \theta, v^i/x \rangle \quad (9)$$
$$\mid \langle \theta, e^i/x \rangle$$

Substitutions

ΓP
purify contexts

$$(\cdot)^P = \cdot$$

$$(\Gamma, \underline{x:Ptr})^P = \Gamma P$$

$$(\Gamma, x:A)^P = \Gamma P, x:A$$

$\Gamma \vdash \theta : \Delta$

$$\frac{\Gamma \vdash \theta : \Delta}{\Gamma \vdash \langle \rangle : \cdot}$$

$\Gamma \vdash \theta : \Delta \quad \Delta \vdash v : A$

$$\frac{\Gamma \vdash \theta : \Delta \quad \Delta \vdash v : A}{\Gamma \vdash \langle \theta, v^i/x \rangle : \Delta, x:A}$$

$\Gamma \vdash \theta : \Delta \quad \Delta^P \vdash e : A$

$$\frac{\Gamma \vdash \theta : \Delta \quad \Delta^P \vdash e : A}{\Gamma \vdash \langle \theta, e^i/x \rangle : \Delta, x:A}$$

Γ, P

$$\frac{}{\Gamma \vdash * : \perp} \text{!-I}$$

$$\frac{\Gamma, x:A \vdash e : B}{\Gamma \vdash \lambda x:A. e : A \Rightarrow B} \lambda I$$

$$\frac{\Gamma \vdash p : \text{Ch} \quad \Gamma \vdash s : \text{Str}}{\Gamma \vdash \text{print}(p, s) : \perp} \text{PRINT}$$

$$\frac{\Gamma, P \vdash e : A}{\Gamma \vdash \text{box}(e) : \Box A} \Box I$$

$$\frac{\Gamma \vdash e_1 : \Box A \quad \Gamma, x:A \vdash e_2 : B}{\Gamma \vdash \text{let box}(x) = e_1 \text{ in } e_2 : B} \Box E$$

This holds

$$\Gamma^P \vdash e = * : \perp$$

$$\Gamma^P \vdash f : A \Rightarrow B$$

$$\Gamma^P \vdash f = \lambda x. f x : A \Rightarrow B$$

⋮
⋮
⋮
 η laws

Translation of pure STLC

(11)

$$(A \Rightarrow B)^{\circ} = \Box A^{\circ} \rightarrow B^{\circ}$$

$$\Gamma \vdash e_1 = e_2 : A \text{ in pure STLC}$$

$$\Gamma^{\circ} \vdash e_1^{\circ} = e_2^{\circ} : A^{\circ} \text{ in } \lambda^{\text{ch}}$$