SPLV Lecture 4

– STLC & CCCs :

every cc poset, $\mathrm{Down}(P) \cong \mathrm{Pos}(P^{op}, 2)$

$\mathrm{PSh}(\mathcal{C}) \cong [\mathcal{C}^{op}, \mathrm{Set}]$

$V\text{-}\mathrm{PSh}(\mathcal{C}) \cong [\mathcal{C}^{op}, V]$

(1) $\underline{\downarrow} : P \to \mathrm{Down}(P)$ $\qquad \underline{\downarrow} : \underline{\mathcal{C}} \to [\mathcal{C}^{op}, \mathrm{Set}]$

$p \mapsto \downarrow(p)$ $\qquad x \mapsto \mathcal{C}(-, x)$

$S \in \mathrm{Down}(P)$ $\qquad P \in \mathrm{PSh}(\mathcal{C})$,

$\cong \bigcup_{s \in S} \downarrow(s)$ $\qquad \cong \int_{c} \downarrow(c)$

(2) $\mathrm{Sh}(\mathcal{C}, \mathcal{J}) \subseteq \mathrm{PSh}(\mathcal{C})$ $\qquad$ <u>f.f.</u>

(3) V-Cat $\qquad V = P(L)$

$V = \Omega$, Pos,

<u>exercise</u>

2

- Linear logic models.

Rel

$A \nrightarrow B$

$!(A \times B) \multimap C$

$\circlearrowright$ $!$

Seely comonad

$\otimes, \multimap$

Seely iso

$\cong \; !(A) \otimes !(B) \multimap C$

$\cong \; !A \multimap (!B \multimap C)$

$\circlearrowright$ (Coh, lin)

$!$ (Coh, stable)

$A \Rightarrow B \rightsquigarrow \underline{!A \multimap B}$

CBN Girard translation.

- effects

$$\frac{}{\Gamma \vdash \mathrm{print} : \mathbb{1}}$$

$$\frac{\Gamma \vdash f : \mathbb{1}}{\Gamma \vdash f = * : \mathbb{1}}$$

$\mathrm{print} \neq *$.

- CCC - a $\mathcal{C}$, (1) $\mathcal{C}(\Gamma, 1) \cong 1$

(2) $\mathcal{C}(\Gamma, A \times B) \cong \mathcal{C}(\Gamma, A) \times \mathcal{C}(\Gamma, B)$

(3) $\mathcal{C}(\Gamma, A \Rightarrow B) \cong \mathcal{C}(\Gamma \times A, B)$.

- partial CCC, $\begin{tikzcd} \mathcal{C} \arrow[loop above, "T"] \end{tikzcd}$ (1), (2) + Partial exponentials.

Kleisli

$v : \mathcal{C} \to \mathcal{C}_T$

$k : \mathcal{C}_T \to \mathcal{C}$

$X \mapsto TX$ strong monad,

(3*) $\cancel{\text{[illegible]}}$

$\mathcal{C}(\Gamma, \underbrace{A \to TB}_{\text{partial exp.}}) \cong \mathcal{C}_T(\Gamma \times A, B)$

values $\begin{tikzcd} \mathcal{C} \arrow[rr, bend right, "v"'] \arrow[r, phantom, "T"] & & \mathcal{C}_T \arrow[ll, bend right, "k"'] \end{tikzcd}$ computations

$v \dashv k$

$\mathcal{C}_T(A, B) = \mathcal{C}(A, TB)$

④

$\square \quad G \quad e \quad T$ fancy $\underline{e}$

$\underline{\square, T, + \text{ interaction}}$

$$\frac{\Gamma \vdash p : FilePtr \qquad \Gamma \vdash s : Str}{\Gamma \vdash print(p, s) : \mathbb{1}}$$

– if $p : FilePtr \in \Gamma$,

$\Gamma \vdash e : \mathbb{1}$,

$\Gamma \supseteq \Delta$, $\underline{\Delta \vdash e : A}$, then $\Gamma \vdash e : A$

– if $p : FilePtr \notin \Gamma$,

$\Gamma \vdash e = * : \mathbb{1}$

Δ —

$\vdots$

$p \cdot \rightsquigarrow \quad \cup\!| \quad$ — $\Gamma', p : Ptr$

Γ

Fix set of names $\mathcal{C}$,

$\mathcal{C}^{\mathrm{cn}}$ $\quad \mathrm{Obj} := \left(A : \mathrm{Set},\ \Vdash \ \subseteq \mathcal{C}_n \times A \right).$

$\Vdash \ \subseteq C \times A$

$\cong w_A : A \to \wp(C)$

$C \times A \to 2$

$\cong A \to 2^{C}$.

$\mathrm{Hom}\left((A, w_A), (B, w_B)\right) \quad w_A : A \to \wp(\mathcal{C}_n)$

$= \{ f : A \to B \mid \forall a \in A,\ w_B(f(a)) \subseteq w_A(a) \}$

$\mathrm{Fam}(\mathrm{Set}^{\mathrm{op}})$

containers.

Kripke sheaves.

$(\wp(\mathcal{C}_n), \supseteq)$

$\mathrm{Fam}(\wp(\mathcal{C}_n), \supseteq)$

$\forall$ complete Heyting alg H.

$\alpha : H \longrightarrow \mathrm{Fam}(H)$

exercise.

- $\mathcal{C}^c$ has products, exponentials, coproducts.

(*) $(A, w_A) \times (B, w_B)$

$= \left(A \times B,\ w_{A\times B}(a,b) = w_A(a) \cup w_B(b)\right)$

(*) $(A, w_A) \Rightarrow (B, w_B)$ (point wise)

exercise

(*) coproducts (free).

- STLC + coproducts.

$(0, 1, +, \times, \Rightarrow)$

- $T : \mathcal{C}^{ch} \longrightarrow \mathcal{C}^{ch}$. (strong monad)

$$(X, w_X) \longmapsto (X \times (Ch \to Str), w_{T(X)})$$

$$w_{TX} : (X \times (Ch \to Str)) \to P(Ch)$$
$$(x, o) \longmapsto w_X(x) \cup \{c \mid o(c) \neq \varepsilon\}.$$

$Ch \in \mathcal{C}^{ch}$

Ch, $w_{Ch} : Ch \to P(Ch)$
$x \mapsto \{x\}$.

- $\Box : \mathcal{C}^{ch} \longrightarrow \mathcal{C}^{ch}$ (strong monoidal comonad)

$$(X, w_X) \longmapsto (\{x \in X \mid w_X(x) = \phi\}, w_{\Box X})$$

- $\phi : \Box T(A) \xrightarrow{\sim} \Box A$

$\varepsilon_A : \Box A \longrightarrow A$

$\delta_A : \Box A \xrightarrow{\sim} \Box^2 A$

$\Box(A \times B) \cong \Box A \times \Box B$.

$$T \circlearrowright \mathcal{C}^{ch}$$

$$\mathcal{C} \circlearrowright \square$$

$$\mathcal{C}^{\square}_{T}(A, B) = \mathcal{C}^{ch}(\square A, TB)$$

$$f : \square A \longrightarrow TB \qquad g : \square B \longrightarrow TC$$

$$f;g \equiv \square A \xrightarrow{\sim} \square\square A \xrightarrow{\square f} \square TB \xrightarrow{\phi} \square B \xrightarrow{g} TC .$$

$$\begin{aligned}
&\mathcal{C}^{\square}_{T}(A \times B, C) \\
&\equiv \mathcal{C}(\square(A \times B), TC) \\
&\cong \mathcal{C}^{ch}(\square A \times \square B, TC) \\
&\cong \mathcal{C}^{ch}(\square A, \underbrace{\square B \to TC})
\end{aligned}$$

$$\underline{A \Rightarrow B \rightsquigarrow \square A \to TB}$$

translation.

$\boxed{\Gamma, \quad \square\Gamma}$

9

$$\theta ::= \langle\rangle \mid \langle \theta, v^i/x \rangle \mid \langle \theta, e^p/x \rangle$$

Substitutions

Γ^P

purify contexts

$$(\cdot)^P = \cdot$$

$$(\Gamma, \underline{x : \mathrm{Ptr}})^P = \Gamma^P$$

$$(\Gamma, x : A)^P = \Gamma^P, x : A$$

$\Gamma \vdash \theta : \Delta$

$$\frac{}{\Gamma \vdash \langle\rangle : \cdot}$$

$$\frac{\Gamma \vdash \theta : \Delta \qquad \Delta \vdash v : A}{\Gamma \vdash \langle \theta, v^i/x \rangle : \Delta, x : A}$$

$$\frac{\Gamma \vdash \theta : \Delta \qquad \Delta^P \vdash e : A}{\Gamma \vdash \langle \theta, e^p/x \rangle : \Delta, x : A}$$

$\cancel{\Gamma^p}$

$$\frac{}{\Gamma \vdash * : \mathbb{1}}\ \mathbb{1}\text{-}I \qquad \frac{\Gamma, x :^i A \vdash e : B}{\Gamma \vdash \lambda x :^i A.\, e : A \Rightarrow B}\ \lambda I$$

$$\frac{\Gamma \vdash p : Ch \qquad \Gamma \vdash s : Str}{\Gamma \vdash \mathrm{print}(p, s) : \mathbb{1}}\ \text{PRINT} \qquad \frac{\Gamma^p \vdash e : A}{\Gamma \vdash \mathrm{box}(e) : \Box A}\ \Box I$$

$$\frac{\Gamma \vdash e_1 : \Box A \qquad \Gamma, x :^p A \vdash e_2 : B}{\Gamma \vdash \mathrm{let}\ \mathrm{box}(x) = e_1\ \mathrm{in}\ e_2 : B}\ \Box E.$$

This holds

$$\frac{}{\Gamma^{P} \vdash e = * : \mathbb{1}}$$

$$\frac{\Gamma^{P} \vdash f : A \Rightarrow B}{\Gamma^{P} \vdash f = \lambda x. f x : A \Rightarrow B}$$

$\vdots$

η laws.

Translation of pure STLC.

$$(A \Rightarrow B)^{\circ} = \Box A^{\circ} \to B^{\circ}$$

$$\frac{\Gamma \vdash e_1 = e_2 : A \text{ in pure STLC}}{\Gamma^{\circ} \vdash e_1^{\circ} = e_2^{\circ} : A^{\circ} \text{ in } \lambda^{ch}.}$$