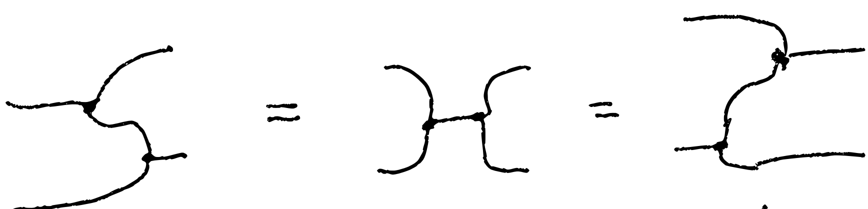
Lecture 3 —
Structured and decorated cospans

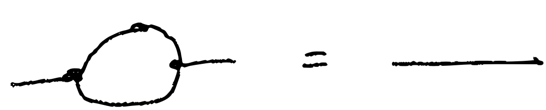
1. Frobenius algebras

A (commutative) <u>Frobenius algebra</u> on an object X in a SMC is a (commutative) monoid + comonoid satisfying the <u>Frobenius law</u>:

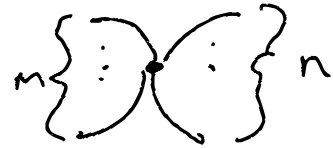


(Mnenomic: preserves connectivity)

The Frobenius algebra is called <u>special</u> if it satisfies

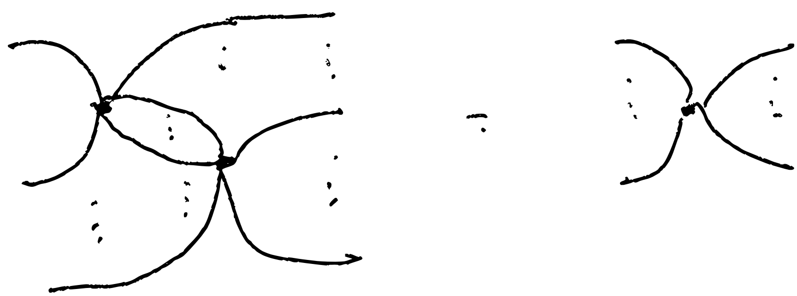


Equivalently: a special commutative Frobenius algebra is given by a family of <u>spiders</u>



($m \geq 0$, $n \geq 0$)

satisfying <u>spider fusion</u>,

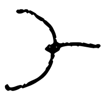


A <u>hypergraph category</u> is a SMC with a supply of commutative Frobenius algebras.

Example

$(\mathrm{Rel}, \otimes)$ is a hypergraph cat with

 $: 1 \nrightarrow X = \{(*, x) \mid x \in X\}$

 $: X \times X \nrightarrow X = \{((x_1, x_2), x_3) \mid x_1 = x_2 = x_3\}$

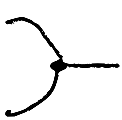
 $: X \nrightarrow 1 = \{(x, *) \mid x \in X\}$

 $: X \nrightarrow X \times X = \{(x_1, (x_2, x_3)) \mid x_1 = x_2 = x_3\}$.

Example

If $\mathcal{C}$ is cartesian then $\mathrm{Span}(\mathcal{C})$ is hypergraph:

 $=$
$$\begin{tikzcd} & X \arrow[dl, "!"'] \arrow[dr, equal] & \\ 1 & & X \end{tikzcd}$$

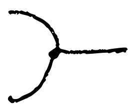
 $=$
$$\begin{tikzcd} & X \arrow[dl, "\Delta"'] \arrow[dr, equal] & \\ X \times X & & X \end{tikzcd}$$

 $=$
$$\begin{tikzcd} & X \arrow[dl, equal] \arrow[dr, "!"] & \\ X & & 1 \end{tikzcd}$$

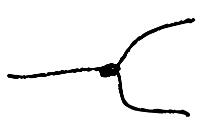
 $=$
$$\begin{tikzcd} & X \arrow[dl, equal] \arrow[dr, "\Delta"] & \\ X & & X \times X \end{tikzcd}$$

Example If $\mathcal{C}$ is cocartesian then $\mathrm{Cospan}(\mathcal{C})$ is hypergraph with $+$:

 $=$ $$\begin{tikzcd} \emptyset \arrow[r, "i"] & X & X \arrow[l, equal] \end{tikzcd}$$

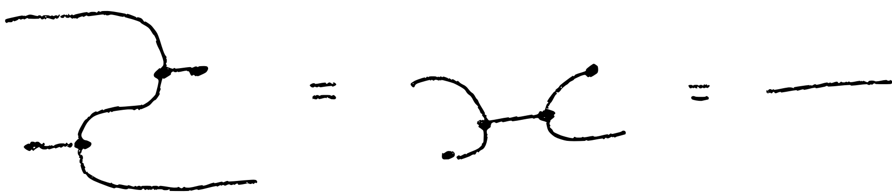
 $=$ $$\begin{tikzcd} X+X \arrow[r, "\nabla"] & X & X \arrow[l, equal] \end{tikzcd}$$

 $=$ $$\begin{tikzcd} X \arrow[r, equal] & X & \emptyset \arrow[l, "i"'] \end{tikzcd}$$

 $=$ $$\begin{tikzcd} X \arrow[r, equal] & X & X+X \arrow[l, "\nabla"'] \end{tikzcd}$$

Theorem If C is hypergraph then it is self-dual compact closed, with

because



2. Structured cospans

Gph is the category of directed graphs + homomorphisms.

It's the category of functors

or FinSet

$$G : \left\{ \begin{tikzcd} E \bullet \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \bullet V \end{tikzcd} \right\} \longrightarrow \mathrm{Set}$$

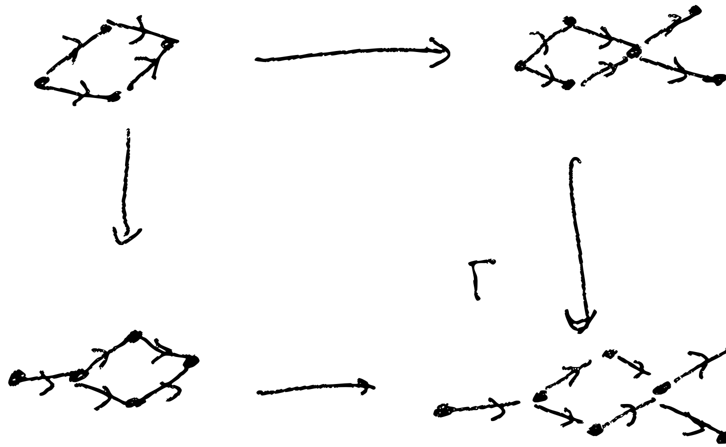
so it's complete, cocomplete, etc.

(it's a topos).

Coproduct in Gph is disjoint union of graphs.

Pushouts in Gph give <u>glueing</u>

of graphs along subgraphs :



⤳ This leads to double pushout (DPO)

graph rewriting.

A *structured cospan* in Gph is a cospan of the form

$$
\begin{tikzcd}
 & G & \\
FX \arrow[ur, "l"] & & FY \arrow[ul, "r"']
\end{tikzcd}
$$

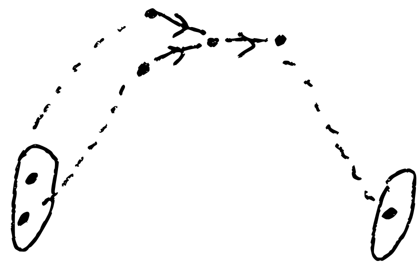
where $F: \mathrm{Set} \longrightarrow \mathrm{Gph}$ is the free graph functor, giving a graph with no edges.

Since F is left adjoint to $V: \mathrm{Gph} \longrightarrow \mathrm{Set}$, F preserves colimits, so we get a monoidal product $\oplus$ where

$$
\begin{tikzcd}
 & G & \\
FX \arrow[ur] & & FY \arrow[ul]
\end{tikzcd}
\oplus
\begin{tikzcd}
 & G' & \\
FX' \arrow[ur] & & FY' \arrow[ul]
\end{tikzcd}
$$

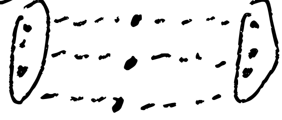
$$
=
\begin{tikzcd}
 & G+G' & \\
F(X+X') \cong FX+FX' \arrow[ur] & & FY+FY' \cong F(Y+Y'). \arrow[ul]
\end{tikzcd}
$$

A structured cospan represents an open graph with a left + right boundary:



There is a category OGph whose objects are sets + morphisms are structured cospans — it's a full subcategory of Cospan(Gph).

Identity morphisms in OGph identify the left + right boundary:



OGph is hypergraph — structure is inherited from Cospan(Gph).

In general, given an adjunction

$$\begin{tikzcd} C \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & D \arrow[l, bend left, "R"] \end{tikzcd},$$

a structured cospan in D is a cospan with feet in the image of L.

3. Semantic functors

Typically we have a description of systems and we would like to compute some kind of semantics : behaviours, solutions etc.

Often, we can get a compositional description of open systems as morphisms of a SMC (so ordinary closed systems are morphisms $I \to I$).

Best case scenario is we get another semantic category $\mathcal{D}$, and a strong monoidal functor $[\![-]\!] : \mathcal{C} \to \mathcal{D}$.

Then we get a divide-and-conquer algorithm for semantics: decompose in $\mathcal{C}$, apply $[\![-]\!]$, compose in $\mathcal{D}$.

Usually this is too good to be true.

— we get an "obvious" semantic category

+ functor, but it fails to be a functor because of "emergent effects"

— a composite can have behaviours that don't arise from behaviours of the parts.

Non-example

Let OGph be the structured cospan category of open graphs.

We care about reachability — which left boundary nodes can reach which right boundary nodes?

We could take the semantic category Span(Set), or alternatively Rel.

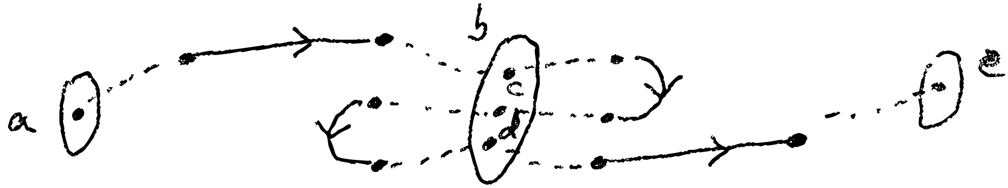
Define $[\![-]\!] : \mathrm{OGph} \longrightarrow \mathrm{Rel}$

by $[\![X]\!] = X$ on objects,

$$\left[\!\!\left[FX \xrightarrow{l} G \xleftarrow{r} FY \right]\!\!\right] = \left\{ (x,y) \;\middle|\; \begin{array}{l} \exists \text{ path} \\ l(x) \rightsquigarrow r(y) \text{ in } G \end{array} \right\}.$$

$[\![-]\!]$ is not a functor! The minimal counterexample is

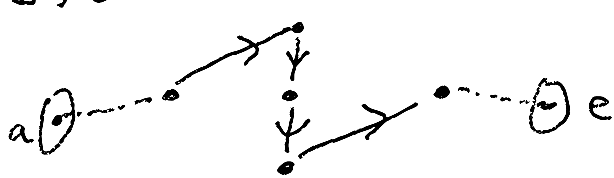
$$1 \xrightarrow{G} 3 \xrightarrow{H} 1,$$



$[\![G]\!] = \{(a,b)\}$, $\quad [\![H]\!] = \{(d,e)\}$

so $[\![G]\!] \,;\, [\![H]\!] = \emptyset$,

but $G \,;\, H =$



so $[\![G \,;\, H]\!] = \{(a,e)\}$.

But we can solve a simplified problem:

Let $\mathrm{OGph}_{\to}$ be the wide subcategory of

$$\begin{tikzcd} & G & \\ FX \arrow[ur, "l"] & & FY \arrow[ul, "r"'] \end{tikzcd}$$

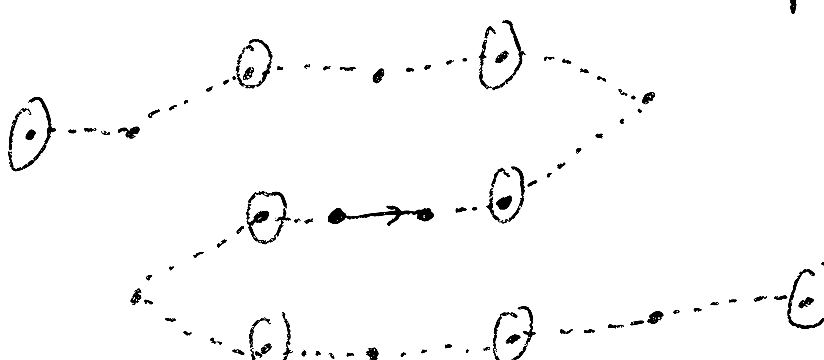
such that

(1) nodes $l(x)$ have no incoming edges,

& (2) nodes $r(y)$ have no outgoing edges.

Then $[\![-]\!]$ restricts to a strong monoidal functor $\mathrm{OGph}_{\to} \to \mathrm{Rel}$.

(This takes care to define $\mathrm{OGph}_{\to}$ correctly — it's not compact closed)



Big example – electrical circuits

A *(resistor) circuit* is an undirected (multi-) graph with an edge labelling from $\mathbb{R}^+ = (0, \infty)$.

There is a category Circ whose objects are circuits, morphisms are label-preserving homomorphisms.

There is an adjunction

$$\begin{tikzcd} \mathrm{Set} \arrow[r, bend left, "\text{discrete}"] \arrow[r, phantom, "\perp"] & \mathrm{Circ} \arrow[l, bend left, "\text{vertices}"] \end{tikzcd}$$

Let OCirc be the structured cospan category.

A <u>linear relation</u> between vector spaces V, W is a linear subspace of $V \oplus W$

"direct product" = biproduct = underlying product.

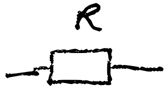
LinRel_K = category of K-vector spaces & linear relations is a hypergraph category.

We can define a semantic functor

$$[\![-]\!] : \mathrm{OCirc} \longrightarrow \mathrm{LinRel}_{\mathbb{R}}$$

idea: each wire goes to the pair (I, V) of current & voltage in that wire.

$$[\![X]\!] = \mathbb{R}^{2 \cdot |X|} = \bigoplus_{x \in X} \mathbb{R}^2$$

$[\![$  $]\!] : \mathbb{R}^2 \nrightarrow \mathbb{R}^2$

$$= \{ (V_1, I_1, V_2, I_2) \mid I_1 = I_2 ,\ V_2 - V_1 = RI \}$$

$[\![$  $]\!] : \mathbb{R}^2 \nrightarrow \mathbb{R}^4$

$$= \{ (V_1, I_1, V_2, I_2, V_3, I_3) \mid I_1 = I_2 + I_3,\ V_1 = V_2 = V_3 \}.$$

$[\![$  $]\!] : \mathbb{R}^2 \nrightarrow \mathbb{R}^0$

$$= \{ (V, I) \mid I = 0 \}$$

This defines a <u>hypergraph functor</u>

$\mathrm{Circ} \to \mathrm{LinRel}_{\mathbb{R}}$

– i.e. strong monoidal + preserves Frobenius structure.

4. Decorated cospans

This is a more complicated but more flexible way to build categories of open systems.

A <u>decorated cospan</u> in $\mathcal{C}$ is a cospan

$$\begin{tikzcd} & A & \\ X \arrow[ur] & & Y \arrow[ul] \end{tikzcd}$$

plus a decoration $d \in FA$

where $F : \mathcal{C} \longrightarrow \mathrm{Set}$.

Need F to be lax monoidal $(\mathcal{C}, +) \longrightarrow (\mathrm{Set}, \times)$ so we have $\nabla : FX \times FY \longrightarrow F(X+Y)$.

To compose:

$$\begin{tikzcd} & & A+_Y B & & A+B \arrow[ll, dashed, "\text{universal map}"'] \\ & A \arrow[ur] & & B \arrow[ul] & \\ X \arrow[ur] & & Y \arrow[ul] \arrow[ur] & & Z \arrow[ul] \end{tikzcd}$$

$d \in FA$ $\qquad$ $d' \in FB$

$d \nabla e \in F(A+B)$

so have

$[d \nabla e] \in F(A +_Y B)$.

Typical example: FX = set of graphs with vertex set X

$\nabla : FX \times FY \longrightarrow F(X+Y)$

= disjoint union of graphs.

This gives us the same category OGraph.

A common source of these:

$\mathcal{C}$ = FinSet,

FX = set of equation systems in some class, with free variables in X.

example: 1st order ODEs

Now boundaries are "exported" variables, composition is union of eqn systems + variable sharing.