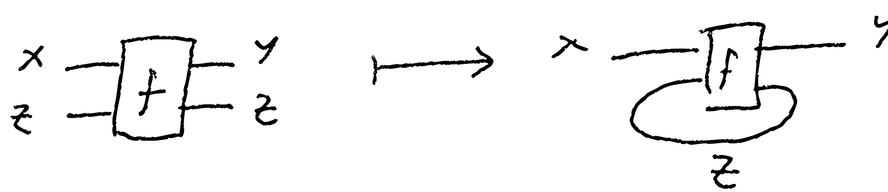
Lecture 2:

Structured monoidal categories.

1. Traces

A symmetric monoidal category is called *traced* if it has an operation

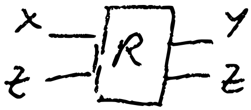


which is isotopy-invariant.
(i.e. a whole bunch of axioms).

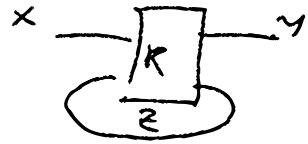
Traced monoidal categories model non-well-founded recursion.

Non-example $(\mathrm{Set}, \times)$ and $(\mathrm{Set}, +)$ do not have a traced structure.

Example $(\mathrm{Rel}, \otimes)$ admits a trace,



$R : X \times Z \nrightarrow Y \times Z$

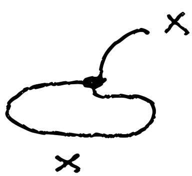


$\mathrm{Tr}\, R : X \nrightarrow Y$

$\{ (x, y) \mid (x, z) R (y, z)$ for some $z \in Z \}$.

Recall $(\mathrm{Rel}, \otimes)$ has a supply of comonoids.

We can calculate :

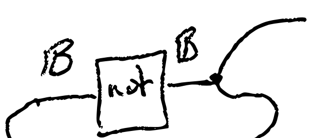


$= \mathrm{Tr}(\Delta) : 1 \nrightarrow X$

$= \{ (*, x) \mid x' \Delta (x, x')$ for some $x' \in X \}$

$= \{ (*, x) \mid x \in X \}$ (full relation)

Another one :



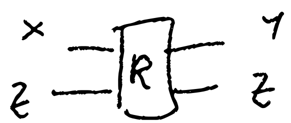
$: 1 \nrightarrow B$

$= \{ (*, b) \mid b = \mathrm{not}\, b \}$

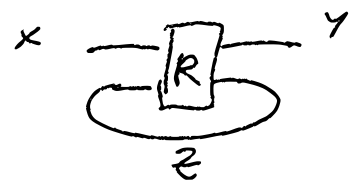
$= \emptyset$ (empty relation)

Idea: Traced monoidal categories must have morphisms that are "paradoxical".

$(\mathrm{Rel}, \oplus)$ is bicartesian, ie.
+ has both co/cartesian universal properties.
It also has a trace:



$\longmapsto$



$R : X + Z \nrightarrow Y + Z$

$R : X \nrightarrow Y$

$= \{ (x, y) \mid$ there is a sequence $z_1, \dots, z_n$ $(n \geq 0)$ st. $x R z_1, z_1 R z_2, \dots, z_{n-1} R z_n, z_n R y \}$.

This looks like operational semantics!

A commutative monoid is

If $\mathcal{C}$ is cocartesian monoidal then it has a supply of monoids,

absurd : $0 \to X$, merge : $X + X \to X$

All morphisms are monoid homomorphisms iff $\mathcal{C}$ is cocartesian monoidal.

Example PFn = category of sets + partial functions.

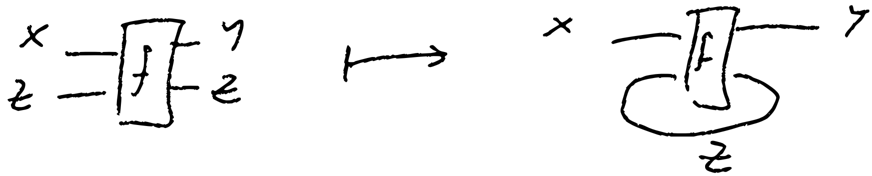
$PFn \cong Kl(1+)$ (aka Maybe monad)

where Nothing ~ nontermination.

$+$ of sets is a coproduct on PFn

(this is true for all kleisli categories)

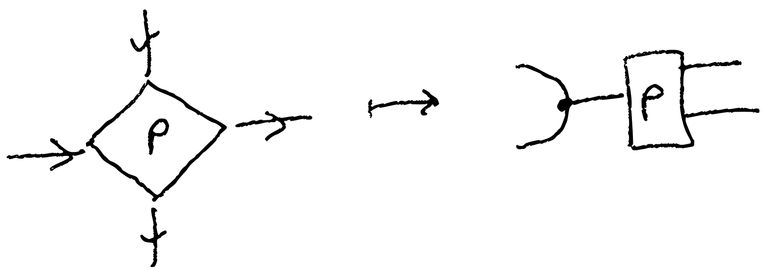
$(PFn, +)$ is traced :



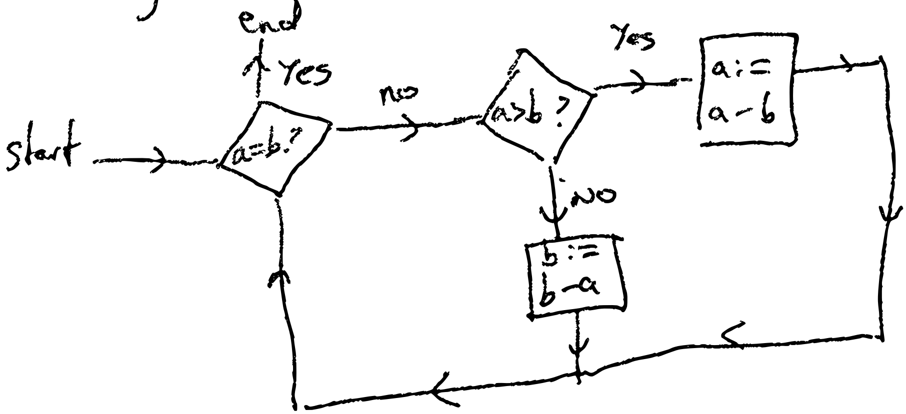
"Keep applying f until it returns a y, if ever"

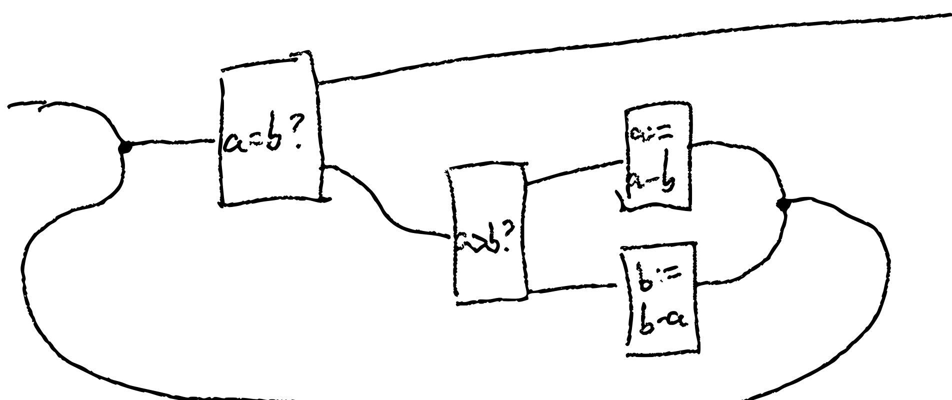
Now string diagrams look like <u>flowcharts</u>!

except we merge incoming connections:



Example Euclid's algorithm for greatest common divisor:





Here all wires have type $X = \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$

$$a=b? : X \longrightarrow X + X,$$
$$(a,b) \longmapsto \begin{cases} \iota_L(a,b) & \text{if } a=b \\ \iota_R(a,b) & \text{otherwise} \end{cases}$$

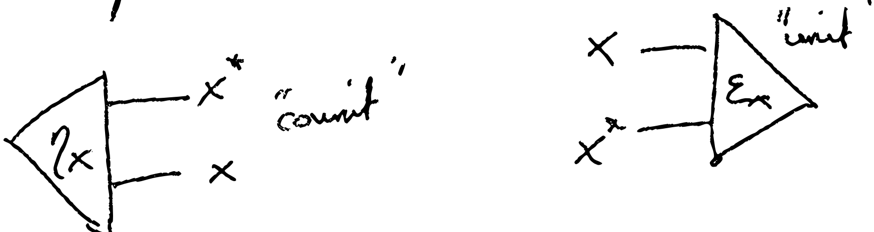
$$a:=a-b : X \longrightarrow X, \ (a,b) \longmapsto (a-b, b)$$

Nb. Euclid's algorithm is guaranteed to terminate, but this perspective can't see that.

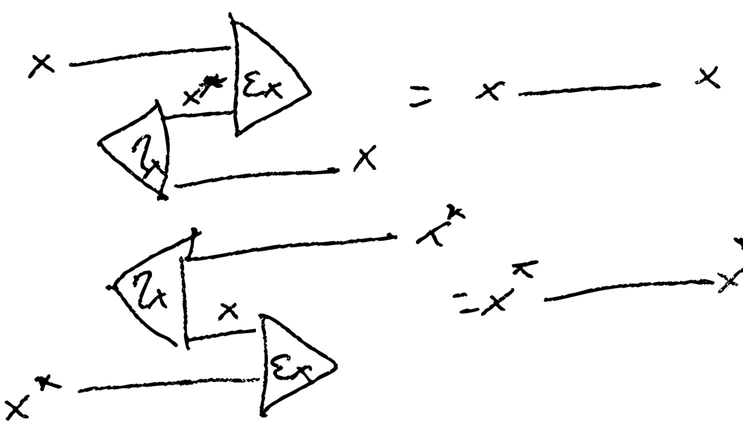
This diagram denotes a morphism $X \to X$ in PFn that happens to be total.

2. Compact closed categories

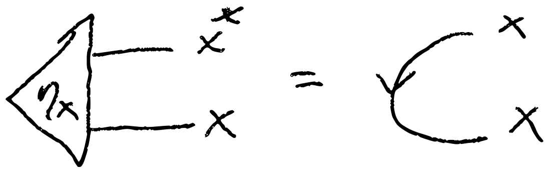
A SMC is called <u>compact closed</u> if for every object X there is an object X^* (s.t. $X^{**} = X$) and morphisms

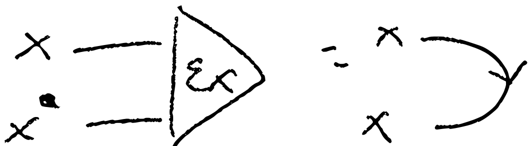


such that

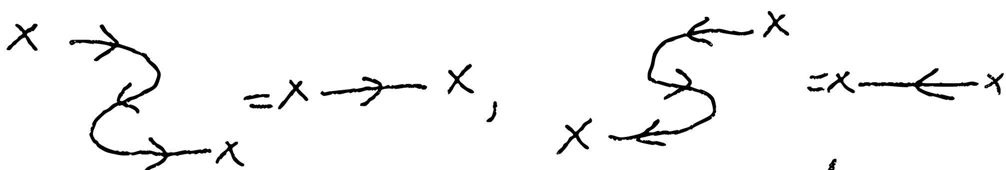


We draw $X^* \longrightarrow X^* = X \longleftarrow X$





so the definition becomes



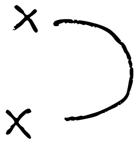
These are called the <u>snake equations</u>.

A compact closed category is <u>self-dual</u>

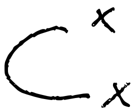
if $X^* = X$

Most CCCs are self-dual in practice.

Example $(\mathbf{Rel}, \otimes)$ is self-dual compact closed:

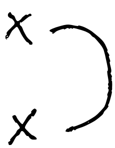


$$: X \times X \nrightarrow 1 = \{((x,x'),*) \mid x = x'\}$$

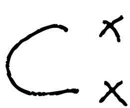


$$: 1 \nrightarrow X \times X = \{(*,(x,x')) \mid x = x'\}.$$

Example $(\mathbf{Span}(\mathbf{Set}), \otimes)$

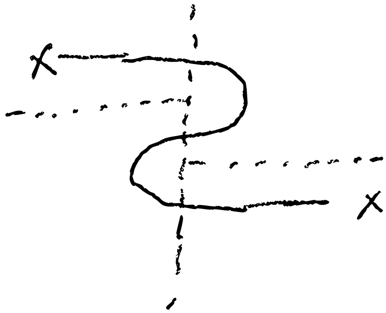


$$= \begin{tikzcd} & X \arrow[dl, "\Delta"'] \arrow[dr, "!"] & \\ X \times X & & 1 \end{tikzcd}$$



$$= \begin{tikzcd} & X \arrow[dl, "!"'] \arrow[dr, "\Delta"] & \\ 1 & & X \times X \end{tikzcd}$$

Let's verify a snake equation:



$\mathrm{id}_X \otimes \eta_X$:

$$\begin{tikzcd} & X \times X \arrow[dl, "\pi_L"'] \arrow[dr, "X \times \Delta"] & \\ X & & X \times X \times X \end{tikzcd}$$

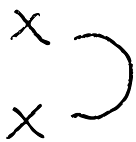
$\varepsilon_X \otimes \mathrm{id}_X$:

$$\begin{tikzcd} & X \times X \arrow[dl, "\Delta \times X"'] \arrow[dr, "\pi_R"] & \\ X \times X \times X & & X \end{tikzcd}$$

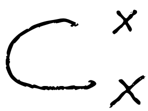
$$\{((x_1, x_2), (x_3, x_4)) \mid (x_1, x_2, x_2) = (x_3, x_3, x_4)\} \cong X$$

$$\begin{tikzcd} & & X \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] \arrow[d, phantom, "\vee"] & & \\ & X \times X \arrow[dl, "\pi_L"'] \arrow[dr, "X \times \Delta"] & {} & X \times X \arrow[dl, "\Delta \times X"'] \arrow[dr, "\pi_R"] & \\ X & & X \times X \times X & & X \end{tikzcd}$$

<u>Example</u> If C is cocartesian then $(\mathrm{Cospan}(\mathcal{C}), \oplus)$ is self-dual compact closed with

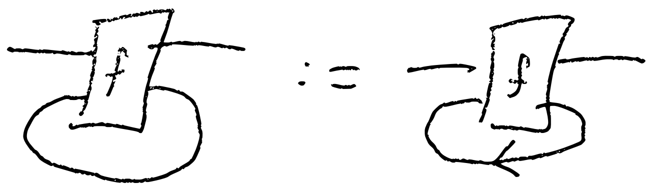


$$= \quad X+X \xrightarrow{\nabla} X \xleftarrow{i} \emptyset$$



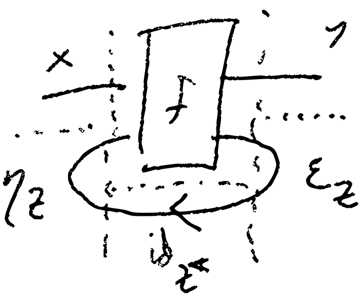
$$= \quad \emptyset \xrightarrow{i} X \xleftarrow{\nabla} X+X$$

Proposition If $\mathcal{C}$ is compact closed then C is traced, with



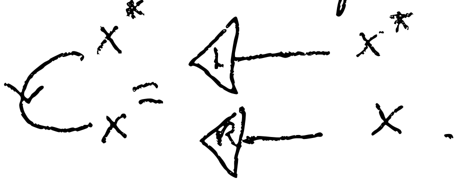
,

i.e.

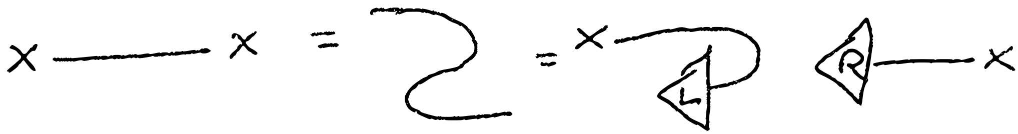


Proposition If $(\mathcal{C}, \otimes)$ is compact closed then $\otimes$ is not cartesian or cocartesian, (unless $\mathcal{C}$ is the terminal category)

Proof idea: In a cartesian category all states are separable, i.e.



Then every identity morphism factors through the terminal object:

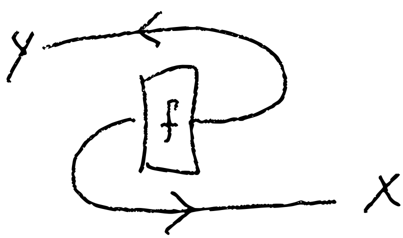


But there is a unique morphism $X \to 1$.
So we can get $X \cong 1$.

Contrast this with traced monoidal categories, which can be co/cartesian.

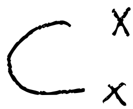
In a compact closed category there is no longer any "causality" from domain to codomain.

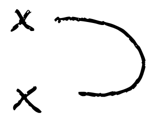
Every morphism $X - \boxed{f} - Y$ has a **transpose** $Y^* - \boxed{f^*} - X^*$ defined by



In $(\mathrm{Rel}, \otimes)$, $(\mathrm{Span}, \otimes)$, $(\mathrm{Cospan}, \oplus)$ transposes are converse relation etc.

We can give (Rel, $\otimes$) a backtracking semantics a la Prolog:

 $= * \mapsto$ do checkpoint;
$x \leftarrow$ arbitrary;
return (x, x)

 $= (x, x') \mapsto$ if $x = x'$
then return $()$
else backtrack

So $f^* : y \mapsto$ do checkpoint;
$x \leftarrow$ arbitrary;
if $y = f(x)$
then return x
else backtrack.

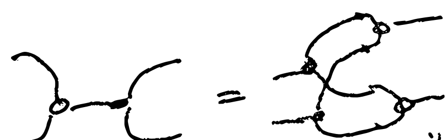
3. Bialgebras

(Maybe this swaps places with
Frobenius algebras from the next lecture)

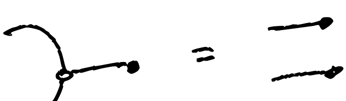
A (commutative) **bialgebra** (terrible name)
or **bimonoid** is an object with a
commutative comonoid 

+ a commutative monoid 

satisfying

(the "bialgebra law")

(mnemonic: number of paths is conserved).

(another mnemonic: each is a
homomorphism of the other).

__Example__ In a cartesian monoidal category, bialgebra = monoid (because copy maps are unique & monoid is deterministic)

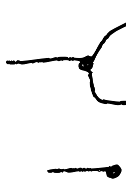
__Example__ If X is a finite (for safety) monoid in Set, then the free vector space $F(X)$ = space of linear combinations from X is a bialgebra

[If X is a group then $F(x)$ is called a __Hopf algebra__, specifically the "group algebra of X" — pure mathematicians love these].

Let $\mathbb{B}$ be the monoid of booleans with $\oplus$ = xor (actually a group)

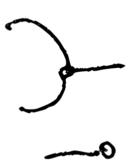
Then $F_{\mathbb{C}}(\mathbb{B}) = \{ \alpha |f\rangle + \beta |T\rangle \mid \alpha, \beta \in \mathbb{C} \}$ is a bialgebra in $(\mathrm{FVect}(\mathbb{C}), \otimes)$

This leads to <u>ZX calculus</u> — an important way of working with <u>quantum circuits</u>. [$F_{\mathbb{C}}(\mathbb{B}) = \mathbb{C}^2$ is the <u>qubit space</u>]



from lifting copy + delete on $\mathbb{B}$: Set to $F_{\mathbb{C}}(\mathbb{B})$

+ transposes



from lifting $\oplus$ + F to $F_{\mathbb{C}}(\mathbb{B})$

+ transposes

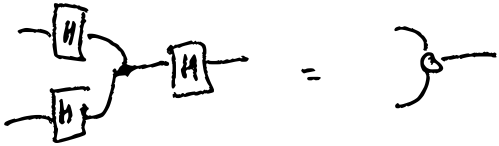
Black + white is a bialgebra
(2 ways)

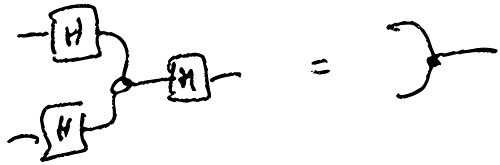
Black + black is a Frobenius algebra
(see next lecture)

white + white is a Frobenius algebra.

& There is a morphism $H: \mathbb{C}^2 \to \mathbb{C}^2$

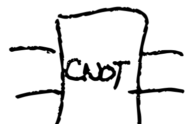
s.t. $H^2 = id$ and called the Hadamard gate





The most important quantum gate is

controlled not



It factors in ZX calculus as