Lecture 1:

Diagrams as syntax

Part 1: Introduction

What is "applied category theory"?

The name is sociological — invented in 2018 to refer to a cluster of existing related ideas.
— it's not an accurate name!

It is interested in <u>processes</u> and <u>systems</u> with a focus on <u>compositionality</u> (CS, physics/engineering, bio..)

modelled often with symmetric monoidal categories, operads & other things that have syntax in <u>diagrams</u>.

In my view — "real" ACT is actually applied —
the trick is to know when to stop doing category theory and do some <u>honest work</u>.

But this course is about <u>theory</u> — ideas that have proven useful for thinking about applications.
(but I'll be less pedantic than a typical category theorist)

This course will be relatively broad & shallow — but focus on commonly recurring structures.

Part 2: String diagrams

Definition A monoidal signature Σ consists of:

1. A set $Ob(\Sigma)$ of object symbols
2. A set $Mor(\Sigma)$ of morphism symbols
3. Functions $s, t : Mor(\Sigma) \to Ob(\Sigma)^*$
   (source, target)

X^* = set of finite lists on X

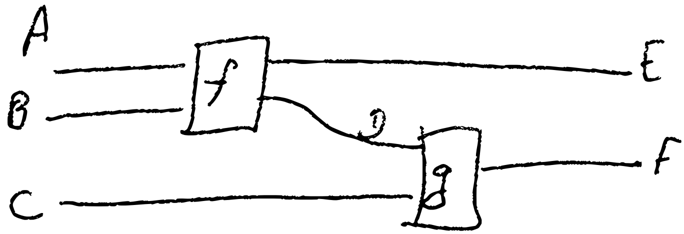
Idea: we think of the list $x_1, \ldots, x_n$ as a formal tensor product $x_1 \otimes \ldots \otimes x_n$

Cf. signatures in logic, Lawvere theories &c.

Note: Monoidal signatures could be called "directed hypergraphs" but it's ambiguous.
Also it's an ordered version of Petri nets.

A <u>string diagram</u> looks like this:

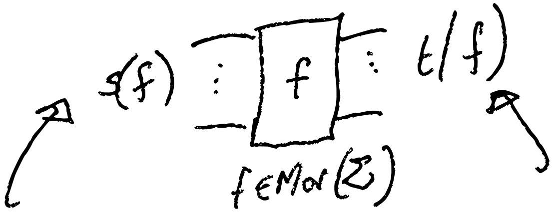
strings/wires
boxes/nodes



[Note: this is left-to-right orientation. Top-to-bottom and bottom-to-top are also in common use].

A string diagram on the signature Σ has:

- strings labelled by object symbols $\in Ob(\Sigma)$
- boxes labelled by morphism symbols $\in Mor(\Sigma)$
- "local connectivity" like this:



$f \in Mor(\Sigma)$

wires listed in bottom-to-top order

If wires do not cross we say the string diagram is <u>planar</u>.

In this course we will allow wires to cross

[Looking ahead: monoidal vs. symmetric monoidal]

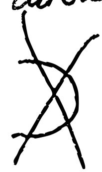
<u>But</u>: The order of connectivity at boxes + diagram boundaries is still important. So:

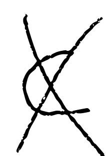
$s(f) = A, B$

but

$s(g) = B, A$

For now, we do not allow wires to change direction:



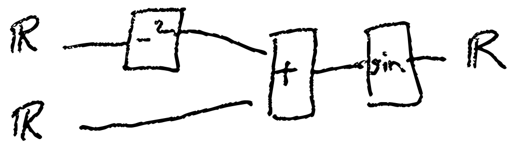


A string diagram with this restriction is called <u>progressive</u>

It's a sort of causality condition.

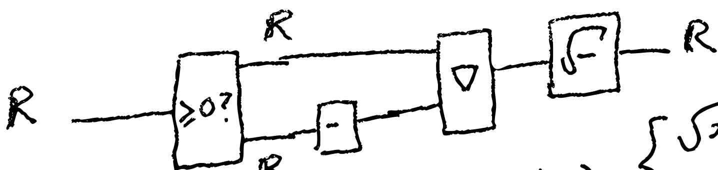
Some monoidal categories of interest

- $(\underline{Set}, \times)$ i.e. total + deterministic functions in a cartesian "wave-style" world



$(x, y) \mapsto \sin(x^2 + y)$

- $(Set, +)$ functions in a "particle-style" world



$$x \mapsto \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ \sqrt{-x} & \text{if } x < 0 \end{cases}$$

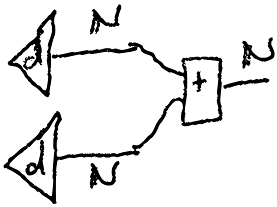
- $(Kl(M), \otimes)$ where M is a $\underline{\text{commutative}}$ monad on Set

i.e. do $\{x \leftarrow a;$ $y \leftarrow b;$ pure $(x,y)\}$ = do $\{y \leftarrow b;$ $x \leftarrow a;$ pure$(x,y)\}$

If M is not commutative then $Kl(M)$ is not monoidal, only premonoidal — possible but annoying to handle

$\lrcorner$

— Let ΔX = set of (finite support) probability distributions on X. Then $Kl(\Delta)$ looks like this:



$d : 1 \to \Delta\mathbb{N}$,
$d(*) = \frac{1}{6}|1\rangle + \cdots + \frac{1}{6}|6\rangle$

$d+d : 1 \to \Delta\mathbb{N}$, $d(*)$ = distribution on sum of 2 dice (i.e. binomial)

This is the prototypical example of a <u>Markov category</u>

— Examples will come back to: linear maps, relations*, spans, cospans

Each of these has a biproduct $\oplus$ arising from $+$, and a tensor product $\otimes$ arising from $\times$.

* $Rel = Kl(P)$ — need to be a bit careful constructively.

Theorem [Joyal-Street coherence theorem]

Let Σ be a monoidal signature,
Let $\mathcal{C}$ be a symmetric monoidal category.

Fix an <u>interpretation</u> of Σ in $\mathcal{C}$, ie.

$[\![-]\!] : ob(\Sigma) \longrightarrow ob(\mathcal{C})$

$[\![-]\!] : Mor(\Sigma) \longrightarrow Mor(\mathcal{C})$

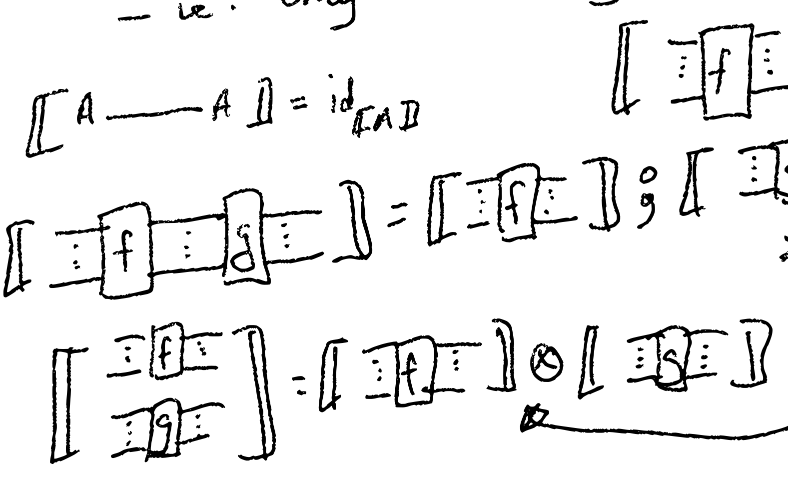
s.t. if $s(f) = x_1, \ldots, x_m$, $t(f) = y_1, \ldots, y_n$

Then $[\![f]\!] : x_1 \otimes \cdots \otimes x_m \to y_1 \otimes \cdots \otimes y_n$

[Note: if $\mathcal{C}$ is not strict monoidal then we have to pick a consistent bracketing].

Then $[\![-]\!]$ can be uniquely extended to string diagrams,
and is <u>isotopy invariant</u>*

– ie. "only connectivity matters".



* when $\mathcal{C}$ is not strict, this case needs more care

Fundamental not-a-theorem of ACT:

Everybody understands the Joyal-Street theorem instinctively!

You can teach it to a 5-year-old

You can teach it to a management consultant

etc.

It's <u>completely intuitive</u> but also <u>completely formal</u>.

The J-S theorem in mathematical context

<u>Thm</u> [J-S, slick version]
Let Σ be a monoidal signature.
Then the free [–, braided, symmetric] monoidal category on Σ is equivalent to the category whose morphisms are planar progressive string diagrams modulo [2d, 3d, 4d] ambient isotopy.

<u>Compare</u>: Let X be a set. Then the following monoids are isomorphic:

1. The free monoid on X
2. The monoid X^*
3. The monoid of "1d string diagrams"

$$-\boxed{x_1}-\boxed{x_2}- \cdots -\boxed{x_n}-$$

modulo 1d isotopy.

And mod 2d isotopy is the free commutative monoid = bags!

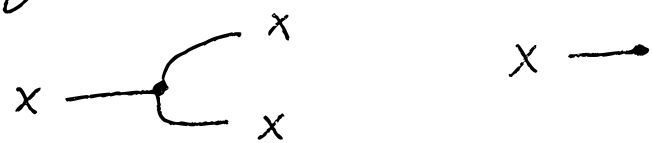
Keywords for more in this direction:

(1) Periodic table of k-tuply monoidal n-categories

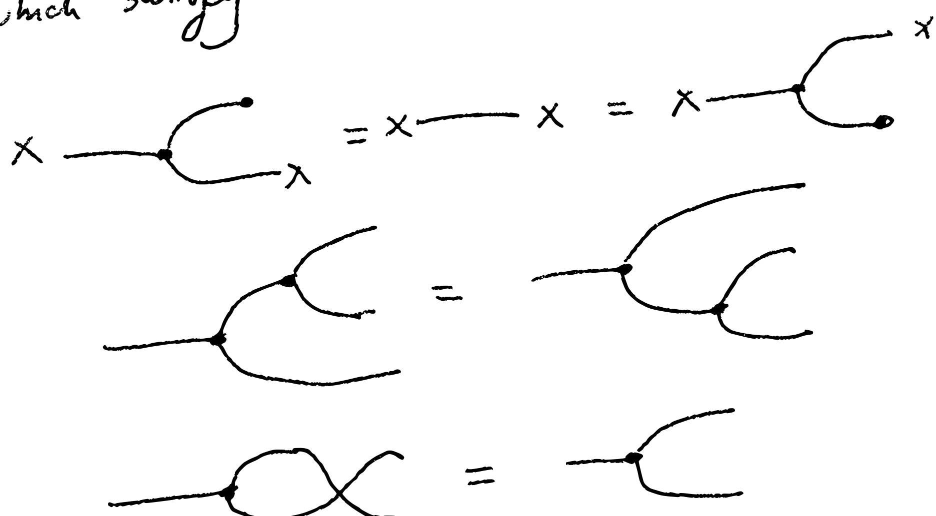
(2) Cobordism hypothesis.

3. Copying and deleting

A (commutative) <u>comonoid</u> on $X \in ob(\Sigma)$ in a string diagram means distinguished nodes

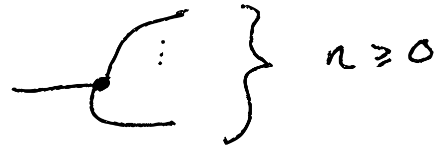


which satisfy

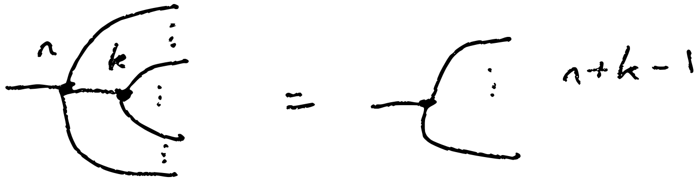


ie. we identify string diagrams up to these equations & require all interpretations satisfy it.

Alternative definition: we have a family of distinguished nodes



satisfying



<u>Examples</u> (lots more in the next 2 lectures)

- In $(\mathrm{Set}, \times)$ or any cartesian monoidal cat we have copy maps
  $\Delta : X \longrightarrow X \times X$, $x \longmapsto (x, x)$
  from the universal property

$$
\begin{tikzcd}
X \arrow[dr, dashed, "\Delta"] \arrow[drr, equal, bend left] \arrow[dd, equal, bend right] & & \\
 & X \times X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & X \\
 & X &
\end{tikzcd}
$$

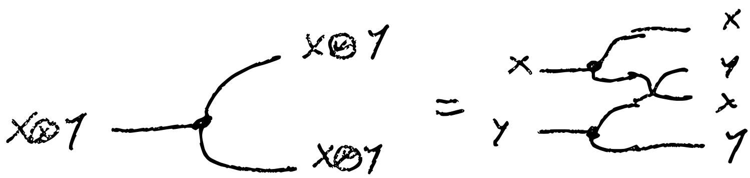
- We can lift that to $(\mathrm{Kl}(M), \otimes)$ by
  $X \xrightarrow{\Delta} X \times X \xrightarrow{\eta_{X \times X}} M(X \times X)$

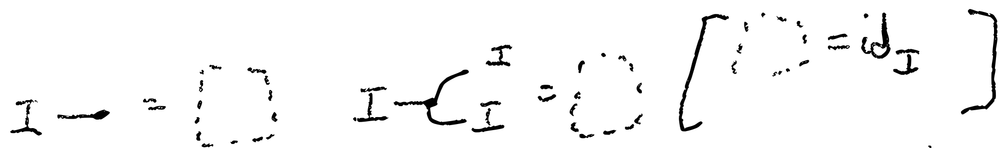
This still satisfies the axioms.

Eg. in $\mathrm{Rel} = \mathrm{Kl}(P)$, $\Delta : X \nrightarrow X \otimes X$,

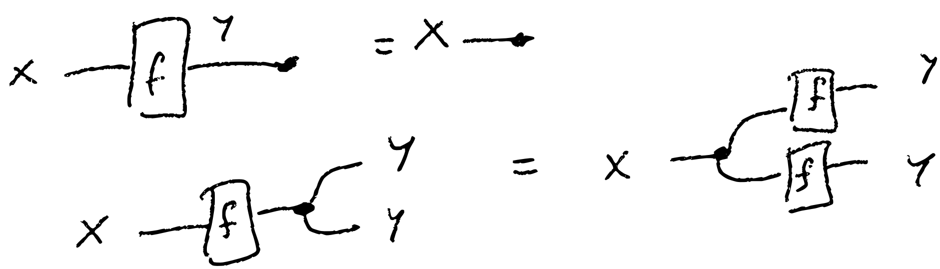
$x_1 \, \Delta \, (x_2, x_3)$ iff $x_1 = x_2 = x_3$.

A <u>supply of comonoids</u> in a sym. monoidal category is a choice of (commutative) comonoid on every object satisfying:





We call a morphism $X \xrightarrow{f} Y$ <u>deterministic</u> if it is a homomorphism of comonoids:



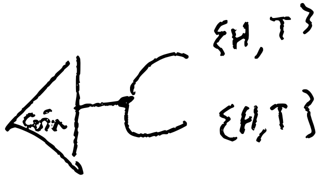
This characterises pure functions!

[Small exercise:  and  are always deterministic].

Example In $Kl(\Delta)$,

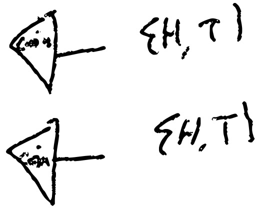
coin : $\underline{1} \to \{H, T\}$, $\text{coin} = \frac{1}{2}|H\rangle + \frac{1}{2}|T\rangle$.

is not deterministic:



$= \frac{1}{2}|(H,H)\rangle + \frac{1}{2}|(T,T)\rangle$

$\neq$



$= \frac{1}{4}|(H,H)\rangle + \frac{1}{4}|(H,T)\rangle + \frac{1}{4}|(T,H)\rangle + \frac{1}{4}|(T,T)\rangle$.

My favourite result in category theory:

Theorem [Fox]

Let $\mathcal{C}$ be a SMC. Then TFAE:

(1) $\mathcal{C}$ admits a supply of commutative comonoids such that all morphisms are deterministic

(2) $\mathcal{C}$ is cartesian monoidal.

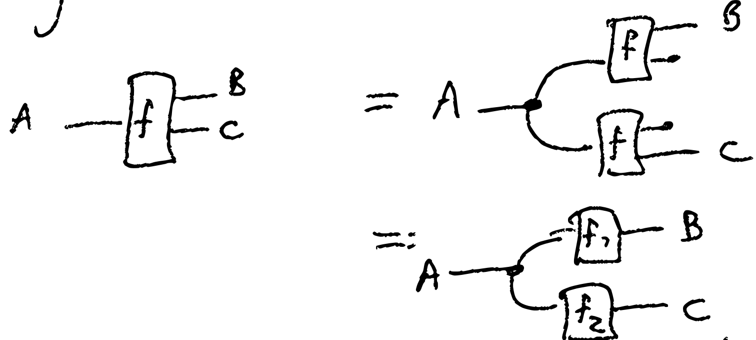
This says we can characterise a universal structure in a purely graphical way!

[Sideline: For any SMC $\mathcal{C}$ there is a category $\mathrm{CComon}(\mathcal{C})$ of comonoids + homomorphisms in $\mathcal{C}$.

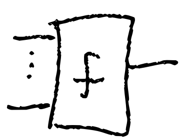
$\mathrm{CComon}(\mathcal{C})$ is cartesian, + is the cofree cartesian cat on $\mathcal{C}$.

Also: $\mathrm{CComon}(\mathcal{K}\ell(M))$ recovers the base category of M].

In a cartesian monoidal category, WLOG we can only consider morphism generators with 1 output, because

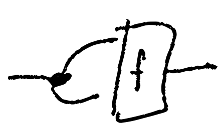


Now graphical syntax is equivalently expressive as traditional term syntax:



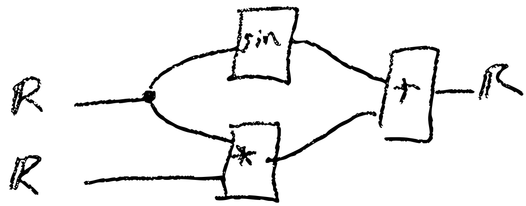
$$\approx \quad f(x_1, \dots, x_n)$$

And copying says we can use variables multiple times:



$$\approx \quad f(x, x)$$

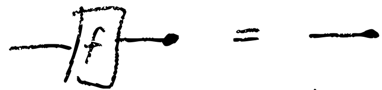
A term written graphically is called a <u>computational graph</u>.



$\simeq \sin x + xy$

They're used in deep learning / differentiable programming — more in lecture 4.

A <u>Markov category</u> is a SMC + supply of commutative comonoids, satisfying



for all morphisms f.

<u>Ex.</u> $Kl(\Delta)$ is a Markov category.

Idea: this is the prototypical one, they are a good axiomatic foundation for probability.