# Type Theory

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(they/them)
Strathclyde
Quantinuum

$$
\begin{aligned}
\sigma, \tau &::= \iota \\
&\mid \sigma \to \tau
\end{aligned}
$$

$$
\begin{aligned}
s, t &::= \lambda x.t \\
&\mid \underline{e}
\end{aligned}
$$

$$
\begin{aligned}
e, f &::= f\ s \\
&\mid x \\
&\mid t : \tau
\end{aligned}
$$

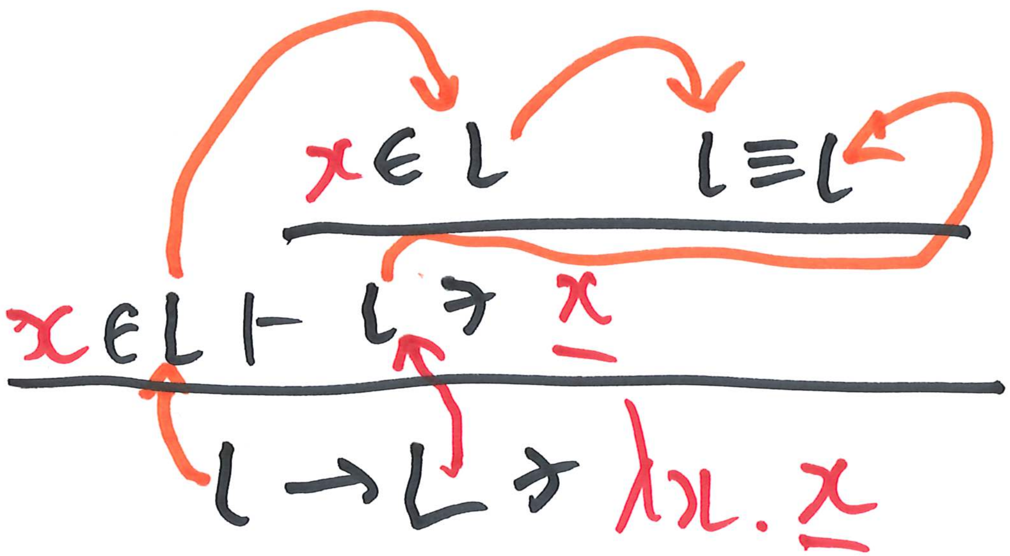
$$\boxed{\tau \ni t} \qquad \boxed{e \in \sigma}$$

$$\frac{x \in \sigma \vdash \tau \ni t}{\sigma \to \tau \ni \lambda x.\, t} \qquad \frac{f \in \sigma \to \tau \quad \sigma \ni s}{f\, s \in \tau}$$

$$\frac{e \in \sigma \quad \sigma \equiv \tau}{\tau \ni \underline{e}} \qquad \frac{\tau \ni t}{t : \tau \in \tau}$$

$$(\lambda x.\, t : \sigma \to \tau)\; s \rightsquigarrow t[s : \sigma] : \tau$$

$$\underline{t : \tau} \rightsquigarrow t$$


x ∈ L    L ≡ L
x ∈ L ⊢ L ∋ x
L → L ∋ λx. x

$$(\lambda x.\lambda y.\ x\,y\,y)\ s\ t$$

$$\rightsquigarrow$$

$$(\lambda y.\ s\,y\,y)\ t$$

$$\rightsquigarrow$$

$$s\,t\,t$$

$$
\begin{aligned}
s, t, S, T &::= \star \\
&\mid \Pi x : S.\, T \\
&\mid \lambda x.\, t \\
&\mid \underline{e}
\end{aligned}
$$

$$
\begin{aligned}
e, f &::= x \\
&\mid f\, s \\
&\mid t : T
\end{aligned}
$$

$$\{\star \ni T\}\ \boxed{T \ni \boxed{t}}\ \{\}$$

$$\{\}\ \boxed{\boxed{e} \in S}\ \{\star \ni S\}$$

$$\frac{}{\star \ni \star}$$

$$\frac{f \in \Pi x{:}S.\,T \qquad S \ni s}{f\ s \in T[s{:}S]}$$

$$\frac{\star \ni S \qquad x \in S \vdash \star \ni T}{\star \ni \Pi x{:}S.\,T}$$

$$\frac{\star \ni T \qquad T \ni t}{t{:}T \in T}$$

$$\frac{x \in S \vdash T \ni t}{\Pi x{:}S.\,T \ni \lambda x.\,t}$$

$$\frac{e \in S \qquad S \equiv T}{T \ni \underline{e}}$$

$$\frac{T \rightsquigarrow T' \qquad T' \ni t}{T \ni t}$$

$$\frac{e \in S \qquad S \rightsquigarrow S'}{e \in S'}$$

$$\underline{t : T} \rightsquigarrow t$$

$$\overbrace{(\lambda x.t : \Pi x{:}S.T)}^{\in\, \Pi x{:}S''.T''}\, s \rightsquigarrow (t : T)[s : S]$$

$$\in T'[s : S']$$

$$\Pi x : \frac{\overbrace{(\lambda X.\underline{X} : \Pi X{:}\star.\star)\,\star}^{\star : \star : \star}}{(\lambda Y.\underline{Y} : \Pi Y{:}\star.\star)\,\star}\ .$$

$$\frac{s \in S \qquad t \in T\xcancel{[s:S]}}{s, t \in \Sigma x : S . T}$$

$$\frac{S \ni s \qquad T[s:S] \ni t}{\Sigma x : S . T \ni s, t}$$

$\boxed{s \triangleright t}$ $\boxed{e \triangleright f}$

$$\frac{t \triangleright t' \quad T \triangleright T'}{x \triangleright x \quad t{:}T \triangleright t'{:}T'}$$

$$\frac{x \vdash t \triangleright t'}{\lambda x.\,t \triangleright \lambda x.\,t'}$$

$$\frac{f \triangleright f' \quad s \triangleright s'}{f s \triangleright f' s'}$$

$$\frac{T \triangleright T' \quad t \triangleright t'}{t{:}T \triangleright t'}$$

$$\frac{x \vdash t \triangleright t' \quad S \triangleright S' \quad x \vdash T \triangleright T' \quad s \triangleright s'}{(\lambda x.\,t : \Pi x{:}S.\,T)\, s \triangleright (t'{:}T')[s'{:}S']}$$

$$x \in S \triangleright x \in S'$$

$$\begin{array}{ccccc} \Gamma & \vdash & T & \ni & t \\ \triangledown^* & & \triangledown^* & & \triangledown \\ \Gamma' & \vdash & T' & \ni & t' \end{array} \qquad \begin{array}{ccccc} \Gamma & \vdash & e & \in & S \\ \triangledown^* & & \triangledown & & \triangledown^* \\ \Gamma' & \vdash & e' & \in & S' \end{array}$$

$$\forall \Gamma, T, t.\ \Gamma \vdash T \ni t \to \forall$$

$$\forall \Gamma, \Gamma', T, T', t, t'.\ \Gamma \vdash T \ni t \wedge \Gamma \triangleright^* \Gamma' \wedge T \triangleright^* T' \wedge t \triangleright t' \longrightarrow \Gamma' \vdash T' \ni t'$$

$$\forall \Gamma, \Gamma', e, e', S.\ \Gamma \vdash e \in S \wedge \Gamma \triangleright^* \Gamma' \wedge e \triangleright e' \longrightarrow \exists S'.\ S \triangleright^* S' \wedge \Gamma' \vdash e' \in S'$$

need $T_2 \leadsto^* T_3$

$T_1 \triangleright^* T_3$

$$\frac{T_0 \leadsto T_1 \qquad T_1 \ni t}{T_0 \ni t}$$

$$\begin{array}{ccc} T_0 & \ni & t \\ \triangledown_* & & \triangledown \\ T_2 & & t' \end{array}$$

$$\frac{e' \in S_2 \rightsquigarrow^{*} S_3 \quad S_1 \rightsquigarrow^{*} S_3 \qquad e \in S_0 \;\Delta^{*}\; S_1 \rightsquigarrow S_1}{\begin{array}{ccc} e & \in & S_1 \\ \nabla & & \nabla^{*} \\ e' & \in & S_3 \end{array}}$$