Category Theory

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Idea:
- like sociology, not neuroscience
- ideal for semantics
- way of thinking
- translate between fields

Def: a **category** consist of
- objects $A, B, C, \ldots$
- morphisms $f, g, h, \ldots : A \longrightarrow B$ for every two objects A, B
- if $f : A \longrightarrow B$ and $g : B \longrightarrow C$, a morphism $g \circ f : A \longrightarrow C$
- for every object A, a morphism $id_A : A \longrightarrow A$

such that:
- associativity: $h \circ (g \circ f) = (h \circ g) \circ f$
- identity: $id_B \circ f = f = f \circ id_A$ for every $f : A \to B$

Examples:

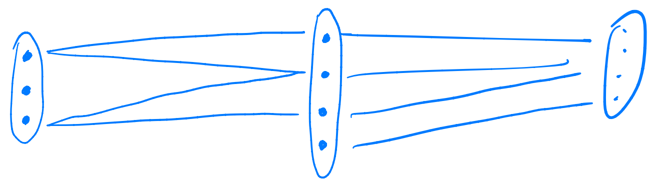
1. Set: obj: sets $A, B, C, \ldots$

   arr: functions $A \xrightarrow{f} B$

2. Rel: obj: sets $A, B, C, \ldots$

   arr: relations $R \subseteq A \times B$

$$A \xrightarrow{R} B \xrightarrow{S = id_C} C \subseteq B$$



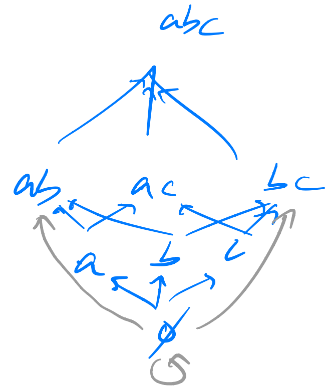
$$S \circ R = \{ (a,c) \in A \times C \mid \exists b \in B : (a,b) \in R \text{ and } (b,c) \in S \}$$

PFn: arr: partial functions $A \to B$

③ if $(P, \leq)$ is partially ordered set

<u>obj</u>: $p \in P$

<u>arr</u>: $p \to q$ iff $p \leq q$



④ Poset: <u>obj</u>: partially ordered set $P, Q, \ldots$

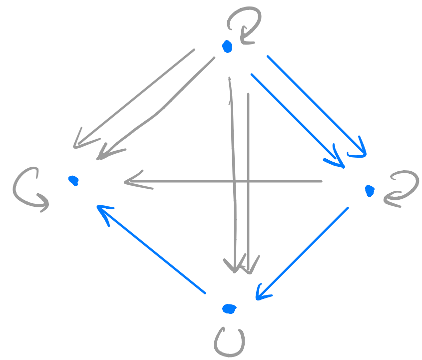
<u>arr</u>: monotone functions $f : P \longrightarrow Q$

$p \leq p' \implies f(p) \leq f(p')$

⑤ if $G=(V,E)$ directed graph

obj: $v \in V$

arr $v \to w$ are paths $v \to w$



⑥ Graph: obj: directed graphs

arr: $(V,E) \to (V',E')$ graph homomorphisms

$f: V \longrightarrow V'$

$(v,u) \in E \Longrightarrow (fv, fu) \in E'$

e.g. $id: V \longrightarrow V$

$v \longmapsto v$

(7) simply typed λ-calculus:

obj: types

arr: $A \to B$ are β-η-equivalence classes of terms of type B with one free var of type A

(8) Hask: obj: Haskell types

arr $A \to B$ are closed Haskell expressions of type $A \to B$

comp: $g \circ f = (\backslash x \to g(f\ x))$

but: $\text{undefined} \circ id = \lambda x.\ \text{undefined} \neq \text{undefined}$

solution? equate $f, g : A \to B$ when $f x = g x$ for all $x : A$ but need operational semantics

Def: if $\mathcal{C}, \mathcal{D}$ are categories, a **functor** $F : \mathcal{C} \longrightarrow \mathcal{D}$ consists of:

- an object $FA \in \mathcal{D}$ for each object $A \in \mathcal{C}$
- a morphisms $Ff : FA \longrightarrow FB$ in $\mathcal{D}$ for each morphism $A \xrightarrow{f} B$ in $\mathcal{C}$

such that:

- $F(g \circ f) = Fg \circ Ff$
- $F(id_A) = id_{FA}$

Examples: ③ if P, Q are posets

and $f : P \longrightarrow Q$ monotone function

regard as categories $\underline{P}$ and $\underline{Q}$

can make functor $\underline{f} : \underline{P} \longrightarrow \underline{Q}$

$$p \longmapsto f(p)$$

$$p \le p' \Longrightarrow f(p) \le f(p')$$

5 If $G=(V,E)$ and $G'=(V',E')$ directed graphs

can regard as categories $\underline{G}$, $\underline{G}'$,

if $f: G \longrightarrow G'$ graph homomorphism,

can regard as functor

$$\begin{aligned} \underline{G} &\longrightarrow \underline{G}' \\ v &\longmapsto f(v) \end{aligned}$$

$$\begin{pmatrix} v_1 \\ \downarrow \\ v_2 \\ \downarrow \\ \vdots \\ \downarrow \\ v_n \end{pmatrix} \longmapsto \begin{pmatrix} f(v_1) \\ \downarrow \\ f(v_2) \\ \downarrow \\ \vdots \\ \downarrow \\ f(v_n) \end{pmatrix}$$

6

$$\begin{aligned}
\mathrm{Set} &\xrightarrow{\Gamma} \mathrm{Rel} \\
A &\longmapsto A \\
(A \xrightarrow{f} B) &\longmapsto \{ (a, f(a)) \in A \times B \mid a \in A \}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Rel} &\xrightarrow{\mathcal{P}} \mathrm{Set} \\
A &\longmapsto \mathcal{P}A \\
(R \subseteq A \times B) &\longmapsto \begin{pmatrix} \mathcal{P}A \longrightarrow \mathcal{P}B \\ U \longmapsto \{ b \in B \mid \exists a \in U : (a,b) \in R \} \end{pmatrix}
\end{aligned}$$

③ $\mathbf{Cat}$: obj : categories $\mathcal{C}, \mathcal{D}, \ldots$

arr functors $\mathcal{C} \to \mathcal{D}$

④ $\mathbf{Posets} \longrightarrow \mathbf{Cat}$

$$P \longmapsto \underline{P}$$

$$f \longmapsto \underline{f}$$

⑥ $\mathbf{Graph} \xrightarrow{\mathrm{Path}} \mathbf{Cat}$

$$G \longmapsto \mathrm{Path}(G)$$

$$f \longmapsto \mathrm{Path}(f)$$

Universal properties

Def: an object $A \in \mathcal{C}$ is **terminal** if
for any object $B \in \mathcal{C}$ there is a (strictly) unique morphism $B \xrightarrow{!=()} A$.

← universal property

Ex:
- Set: any singleton set
- Rel: the empty set
- Poset: any singleton poset

Def: a morphism $f : A \to B$ is an **isomorphism** if
there is $g : B \to A$ such that $g \circ f = id_A$ and $f \circ g = id_B$

Lem: terminal objects are unique up to isomorphism.

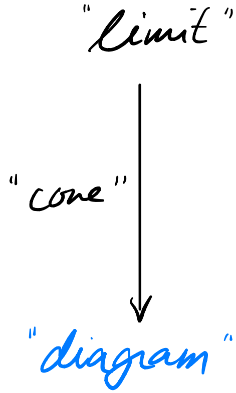
Pf: If A, B terminal, then
$$\begin{tikzcd} A \arrow[loop left, "id_A"] \arrow[r, shift left] & B \arrow[l, shift left] \arrow[loop right, "id_B"] \end{tikzcd}$$

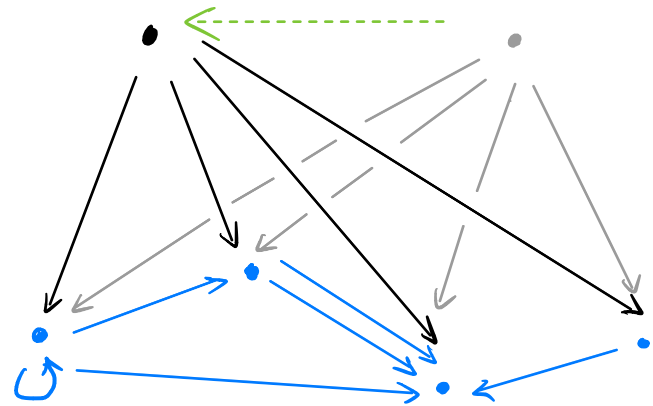
Def: If $A, B \in \mathcal{C}$, a product of A and B is an object $A \times B$ together with map $A \xleftarrow[\pi_A]{fst} A \times B \xrightarrow{snd} B$ such that for every object C and maps $A \xleftarrow{f} C \xrightarrow{g} B$ there is unique map $\langle f,g \rangle : C \longrightarrow A \times B$ s.t. $f = fst \circ \langle f,g \rangle$

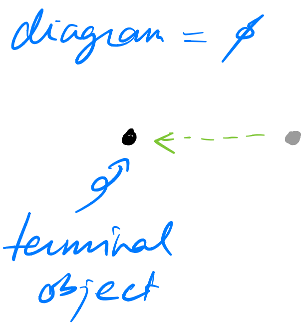
and $g = snd \circ \langle f,g \rangle$

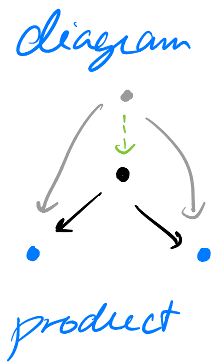
$$\begin{tikzcd}
 & C \arrow[dl, "f"'] \arrow[d, dashed, "{\langle f,g \rangle}"] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "fst"] \arrow[r, "snd"'] & B
\end{tikzcd}$$

Ex:
- Set: cartesian product of sets is a (categorical) product
- Rel: disjoint union of sets is a (categorical) product

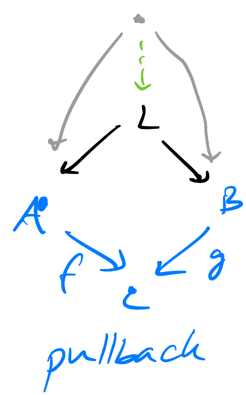








in Set: $L = \{(a,b) \in A \times B \mid fa = gb\}$



Def: If $A, B \in \mathcal{C}$, an exponential object is an object $B^A \in \mathcal{C}$

$$\begin{tikzcd}
C \arrow[d, "f"'] & C \times A \arrow[d, dashed] \arrow[dr, "ev'"] & \\
B^A & B^A \times A \arrow[r, "ev"'] & B
\end{tikzcd}$$

Ex: – in Set, $B^A = \{f : A \to B\}$

– in Graph, $H^G = \{f : (\bullet \to \bullet) \times G \to H \text{ graph homomorphism}\}$

Def: a category is **cartesian closed** if it has a terminal object, all (binary) products, and all exponential objects

if A, B are objects then $A \times B$ exists

and B^A exist

Thm: $\mathbf{Cart} \rightleftarrows \mathbf{Lambda}$

obj: cartesian closed cats

arr: functors

models of simply-typed λ-calculus

$\mathcal{C}at$: obj: categories is cartesian closed:

arr: functors if $\mathcal{C}, \mathcal{D}$ are categories,

$\mathcal{C} \times \mathcal{D}$? obj: (A, B) where $A \in \mathcal{C}$, $B \in \mathcal{D}$

arr $(A,B) \to (A',B')$ are pairs (f,g) where $f : A \to A'$ in $\mathcal{C}$
$g : B \to B'$ in $\mathcal{D}$

$\mathcal{D}^{\mathcal{C}} = [\mathcal{C}, \mathcal{D}]$? obj: functors $F : \mathcal{C} \to \mathcal{D}$

arr $F \to G$: natural transformations, i.e. collection

$\alpha_A : FA \to GA$ for each $A \in \mathcal{C}$ such that

$[\mathcal{C}, Set]$ presheaf

$$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\alpha_B"'] & GB \end{tikzcd}$$

Def 1: a **monad** ("standard construction", "triple") consists of

- a functor $T : \mathcal{C} \to \mathcal{C}$
- a natural transformation $\eta_A : A \to TA$ "unit"

  $\mu_A : T^2 A \to TA$ "multiplication" "join"

such that

$$\begin{tikzcd} TA \arrow[r, "T\eta_A"] \arrow[d, "\eta_{TA}"'] \arrow[dr, equal] & T^2A \arrow[d, "\mu_A"] \\ T^2A \arrow[r, "\mu_A"'] & TA \end{tikzcd}$$

"unit laws"

$$\begin{tikzcd} T^2(TA) = T^3A \arrow[r, "T\mu_A"] \arrow[d, "\mu_{TA}"'] & T^2A \arrow[d, "\mu_A"] \\ T^2A \arrow[r, "\mu_A"'] & TA \end{tikzcd}$$

"associative law"

Def 2: a **monad** ("Kleisli triple") consists of

- an object $TA \in \mathcal{C}$ for every object $A \in \mathcal{C}$ — "type constructor"
- morphisms $\mathrm{return} : A \longrightarrow T(A)$ — "type converter"
- a morphism $\mathrm{bind}(-, f) = (- \gg= f) : TA \longrightarrow TB$ for each $f : A \longrightarrow TB$ — "combinator"

such that:

$$\mathrm{return}(x) \gg= f \quad = \quad f(x)$$

$$t \gg= \mathrm{return} \quad = \quad t \qquad \text{where } "t : TA"$$

$$t \gg= \backslash x \rightarrow (f(x) \gg= g) \quad = \quad (t \gg= f) \gg= g$$

Def 1 = def 2: $\Longrightarrow$: $\mathrm{bind}(x, f) = \mu_{Tf}(x)$ $\qquad \eta = \mathrm{return}$

$\Longleftarrow$: $\mu = \mathrm{bind}(-, \mathrm{id}_{TA})$

Remark: do notation: $\quad$ do $a \leftarrow b \; \ldots \; = \; b \gg= (\backslash a \longrightarrow \ldots)$

Example ① Maybe: $\mathcal{C} = \mathrm{Set}$

$$T(A) = A + 1$$
$$T(f) = f + 1$$

$$\begin{aligned} \eta_A : A &\longrightarrow A+1 \\ a &\longmapsto a \end{aligned}$$

$$\begin{aligned} \mu_A : (A+1)+1 &\longrightarrow A+1 \\ a &\longmapsto a \\ * &\longmapsto * \\ * &\longmapsto * \end{aligned}$$

② Exception: $\mathcal{C} = \mathrm{Set}$ — works in any category with coproducts

$T = (-) + E$ — fixed set of 'exceptions'

$$\eta_A(a) = a$$

$$\begin{aligned} \mu_A : (A+E)+E &\longrightarrow A+E \\ a &\longmapsto a \\ e &\longmapsto e \\ e &\longmapsto e \end{aligned}$$

③ Reader: $TA = A^I$ ~ fixed obj of 'inputs'

$$\frac{A \xrightarrow{\eta_A} A^I}{A \times I \xrightarrow[\mathrm{fst}]{} A}$$

$$\frac{(A^I)^I \xrightarrow{\mu_A} A^I}{(A^I)^I \times I \longrightarrow A}$$

e.g. in Set: $\mu_A(\varphi)(i) = \varphi(i)(i)$

$$\begin{tikzcd}
(A^I)^I \times I \arrow[r] \arrow[d, "id \times \langle id, id \rangle"'] & A \\
(A^I)^I \times (I \times I) \arrow[r, "ev \times id_I"'] & A^I \times I \arrow[u, "ev"']
\end{tikzcd}$$

④ Writer: $TA = A \times O$ ~ fixed object of 'outputs' that is a semigroup

$$O \times O \xrightarrow{c} O \qquad 1 \xrightarrow{e} O$$

$$\begin{tikzcd}
A \arrow[rr, "\eta_A"] \arrow[dr] & & A \times O \\
& A \times 1 \arrow[ur, "id_A \times e"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(A \times O) \times O \arrow[rr, "\mu_A"] \arrow[dr] & & A \times O \\
& A \times (O \times O) \arrow[ur, "id_A \times c"'] &
\end{tikzcd}$$

5. State: $TA = (A \times S)^S$ where S fixed object of 'states'

$$\frac{A \xrightarrow{\eta_A} (A\times S)^S}{A\times S \xrightarrow{id_{A\times S}} A\times S}$$

$$\frac{((A\times S)^S \times S)^S \xrightarrow{\mu_A} (A\times S)^S}{((A\times S)^S \times S)^S \times S \xrightarrow{ev} (A\times S)^S \times S \xrightarrow{ev} A\times S}$$

⑥ Nondeterminism

$\mathcal{C} = \mathrm{Set}$

$TA = \mathcal{P}A$ List A

$$\begin{aligned} A &\xrightarrow{\eta_A} \mathcal{P}A \\ a &\longmapsto \{a\} \end{aligned}$$

$$\begin{aligned} \mathcal{P}\mathcal{P}A &\xrightarrow{\mu_A} \mathcal{P}A \\ \mathcal{A} &\longmapsto \{a \in A \mid \exists U \in \mathcal{A} : a \in U\} \end{aligned}$$

concat

⑦ Probability

$\mathcal{C} = \mathrm{Set}$

$$TA = \{p : A \to [0,1] \mid \mathrm{supp}(p) \text{ finite and } \sum_{a \in A} p(a) = 1\}$$

$$\begin{aligned} \eta_A : A &\longrightarrow TA \\ a &\longmapsto \lambda b : A . \begin{cases} 0 & \text{if } a \neq b \\ 1 & \text{if } a = b \end{cases} \end{aligned}$$

Dirac distribution

$$\begin{aligned} \mu_A : T^2A &\longrightarrow TA \\ P &\longmapsto \lambda a : A . \sum_{p \in TA} P(p) \cdot p(a) \end{aligned}$$

"pure" computations $\Longrightarrow$ "effectful computations"

$\in$ $\qquad\qquad\qquad\qquad$ $\ni$

Prop: If $T : \mathcal{C} \to \mathcal{C}$ is a monad, then there is a category $Kl(T) = \mathcal{C}_T$

with <u>obj</u>: $A \in \mathcal{C}$ same as in $\mathcal{C}$

<u>arr</u> $A \to B$: are morphisms $f : A \to T(B)$ in $\mathcal{C}$

"Kleisli category of T"

Pf: What is composition?

given $A \xrightarrow{f} T(B)$ and $B \xrightarrow{g} T(C)$ in $\mathcal{C}$,

want $A \xrightarrow[f]{} TB \xrightarrow[Tg]{} T^2(C) \xrightarrow[\mu_C]{} T(C)$ in $\mathcal{C}$

What are identities?

want $A \xrightarrow{\eta_A} T(A)$ in $\mathcal{C}$

Ex ① $\mathrm{Kl}(\mathrm{Maybe})$: obj: sets A, B, C

$\quad \|$ arr $A \to B$ are functions $A \longrightarrow B+1$

PFun can be thought of as partial functions $A \longrightarrow B$

② $\mathrm{Kl}(\mathrm{Powerset})$: arr $A \to B$ are functions $A \xrightarrow{f} \mathcal{P}B$

$\quad \|$ can be thought of as relations $R \subseteq A \times B$

Rel

$$R = \{(a,b) \mid b \in f(a)\}$$

$$f(a) = \{b \mid (a,b) \in R\}$$

Def: an (Eilenberg–Moore) algebra of a monad $T : \mathcal{C} \to \mathcal{C}$ consists of:

- an object $A \in \mathcal{C}$ — "carrier object"
- a morphism $TA \xrightarrow{a} A$ — "structure map"

such that

$$\begin{tikzcd} T^2A \arrow[r, "Ta"] \arrow[d, "\mu_A"'] & TA \arrow[d, "a"] \\ TA \arrow[r, "a"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, equal] & TA \arrow[d, "a"] \\ & A \end{tikzcd}$$

Prop: there is a category $EM(T) = \mathcal{C}^T$ with

obj: EM-algebras

arr $(A,a) \to (B,b)$ are $f : A \to B$ in $\mathcal{C}$ s.t.

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "a"'] & TB \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Ex: $EM(\text{Powerset}) = \text{CompleteLattices}$ — obj poset $(P, \leq)$ s.t. each subset has a least upper bound $a(P)$

arr functions that preserve $\bigvee$

Def Functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ are adjoint, F being left adjoint, G right adjoint, denoted $F \dashv G$

① if there is a natural bijection

$$
\begin{tikzcd}
\{ f : FA \arrow[r] & B \arrow[d] & \text{in } \mathcal{D} \} \arrow[r, "\cong", "\varphi_{AB}"'] & \{ g : A \arrow[r] & GB \arrow[d] & \text{in } \mathcal{C} \} \\
FA' \arrow[u] & B' & & A' \arrow[u] & GB' &
\end{tikzcd}
$$

or, equivalently

② there are natural transformations
$\eta_A : A \to GFA$ "unit"
$\varepsilon_B : FGB \to B$ "counit"

satisfying

$$
\begin{tikzcd}
FA \arrow[r, "F\eta_A"] \arrow[dr, equal] & FGFA \arrow[d, "\varepsilon_{FA}"] \\
& FA
\end{tikzcd}
\qquad
\begin{tikzcd}
GA \arrow[d, "\eta_{GA}"'] \arrow[dr, equal] & \\
GFGA \arrow[r, "G\varepsilon_A"'] & GA
\end{tikzcd}
$$

"zig-zag equations"

Ex: ①

$$\mathrm{Set} \underset{\text{Powerset}}{\overset{\Gamma}{\underset{\perp}{\rightleftarrows}}} \mathrm{Rel}$$

$$\frac{\dfrac{R \subseteq A \times B}{\Gamma A \to B \text{ in Rel}}}{\dfrac{A \to \mathcal{P}B \text{ in Set}}{f : A \to \mathcal{P}B}}$$

$f(a) = \{ b \mid (a,b) \in R \}$

$R = \{ (a,b) \mid f(a) \ni b \}$

"free"

Lem: right adjoint preserve limits

$A \mapsto (\mathrm{List}\,A, \text{concat})$

Ex ②:

$$\mathrm{Set} \underset{\text{Forget}}{\overset{\text{List}}{\underset{\perp}{\rightleftarrows}}} \mathrm{Mon}$$

sets A with $A \times A \to A$, $1 \to A$

$A \leftarrow (A, \cdot)$

"forgetful"

Lem: If $\begin{tikzcd} \mathcal{C} \arrow[r, shift left=1.5, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left=1.5, "G"] \end{tikzcd}$ adjoint, then $T = GF$ is monad on $\mathcal{C}$

Thm:

$$\begin{tikzcd}
\mathrm{Kl}(T) \arrow[r, dashed] \arrow[dr, shift left=1.5] \arrow[dr, phantom, "\perp"] & \mathcal{D} \arrow[r, dashed] \arrow[d, shift left=1.5, "G"] \arrow[d, phantom, "\dashv"] & \mathrm{EM}(T) \arrow[dl, shift left=1.5] \arrow[dl, phantom, "\perp"] \\
 & \mathcal{C} \arrow[ul, shift left=1.5] \arrow[u, shift left=1.5, "F"] \arrow[ur, shift left=1.5] \arrow[loop below, "T = GF"] &
\end{tikzcd}$$

Def: a symmetric monoidal category is a category $\mathcal{C}$ with

- a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- a object I
- natural isomorphisms

$$\begin{tikzcd} A \arrow[d, "f"'] & B \arrow[d, "g"] & A \otimes B \arrow[d, "f \otimes g"] \\ A' & B' & A' \otimes B' \end{tikzcd}$$

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \longrightarrow (A \otimes B) \otimes C \quad \text{"associator"}$$

$$\left.\begin{aligned} \lambda_A &: I \otimes A \longrightarrow A \\ \rho_A &: A \otimes I \longrightarrow A \end{aligned}\right\} \text{"unitors"}$$

$$\gamma_{A,B} : A \otimes B \longrightarrow B \otimes A$$

such that

$$\begin{tikzcd} A \otimes (I \otimes B) \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "id_A \otimes \lambda_B"'] & & (A \otimes I) \otimes B \arrow[dl, "\rho_A \otimes id_B"] \\ & A \otimes B & \end{tikzcd}$$

"triangle equations"

$$\begin{tikzcd} & A \otimes ((B \otimes C) \otimes D) \arrow[r, "\alpha_{A, B \otimes C, D}"] & (A \otimes (B \otimes C)) \otimes D \arrow[dr, "\alpha_{A,B,C} \otimes id_D"] & \\ A \otimes (B \otimes (C \otimes D)) \arrow[ur, "id_A \otimes \alpha_{B,C,D}"] \arrow[drr, "\alpha_{A,B,C \otimes D}"'] & & & ((A \otimes B) \otimes C) \otimes D \\ & & (A \otimes B) \otimes (C \otimes D) \arrow[ur, "\alpha_{A \otimes B, C, D}"'] & \end{tikzcd}$$

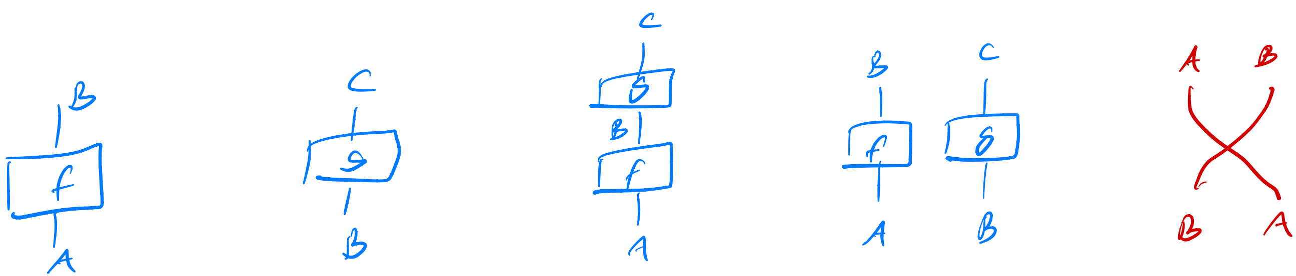
"pentagon equations"

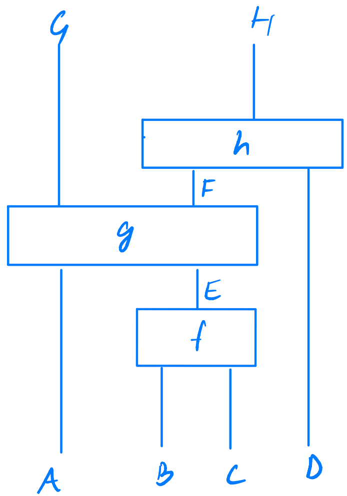
Ex: - If $\mathcal{C}$ has finitary products, then $\otimes = \times$, $I = 1$ makes $\mathcal{C}$ monoidal

- Vect under tensor product

Thm: ("coherence theorem") any two "well-typed" morphisms $A \to B$ in a monoidal category built from $id, \alpha, \lambda, \rho$ using $\circ, \otimes$ are equal

$f : A \to B$ $\quad$ $g : B \to C$ $\quad$ $g \circ f : A \to C$ $\quad$ $f \otimes g : A \otimes B \to B \otimes C$





$\leftrightsquigarrow$

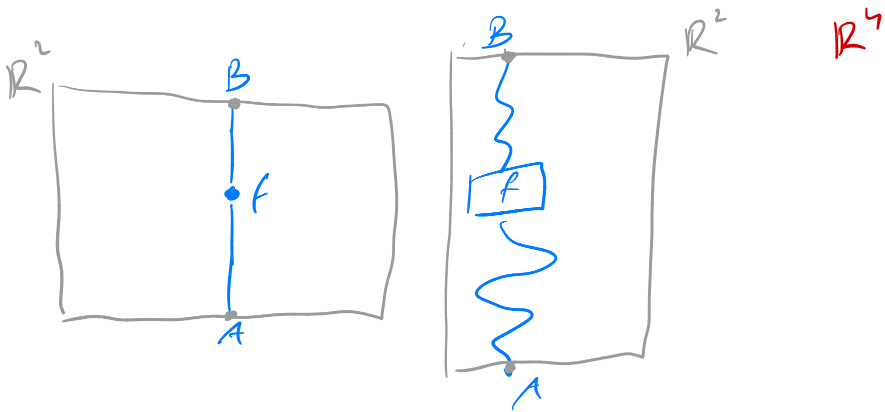
$$
\begin{tikzcd}
G \otimes H \\
G \otimes (F \otimes D) \arrow[u, "\mathrm{id}_G \otimes h"'] \\
(G \otimes F) \otimes D \arrow[u, "\alpha^{-1}_{G,F,H}"'] \\
(A \otimes E) \otimes D \arrow[u, "g \otimes \mathrm{id}_D"'] \\
A \otimes (E \otimes D) \arrow[u, "\alpha_{A,E,D}"'] \\
A \otimes ((B \otimes C) \otimes D) \arrow[u, "\mathrm{id}_A \otimes (f \otimes \mathrm{id}_D)"'] \\
A \otimes (B \otimes (C \otimes D)) \arrow[u, "\mathrm{id}_A \otimes \alpha_{B,C,D}"']
\end{tikzcd}
$$

Thm ("Correctness of graphical calculus")

two morphisms $A \to B$ in a monoidal category are equal

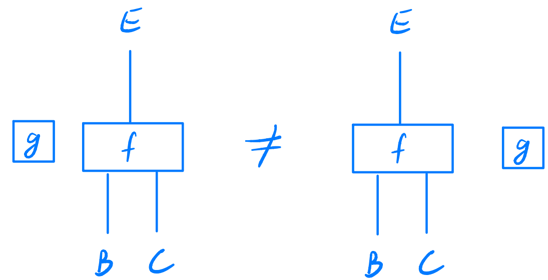
$$\Longleftrightarrow$$

their pictures are isotopic



But e.g.:

$$B\otimes C \xrightarrow{\lambda^{-1}_{B\otimes C}} I\otimes(B\otimes C) \xrightarrow{g\otimes f} I\otimes E \xrightarrow{\lambda_E} E \quad \leftrightsquigarrow$$



$$\leftrightsquigarrow \quad B\otimes C \xrightarrow{\rho^{-1}_{B\otimes C}} (B\otimes C)\otimes I \xrightarrow{f\otimes g} E\otimes I \xrightarrow{\rho_E} E$$

Def: a monad $T: \mathcal{C} \to \mathcal{C}$ on a monoidal category $\mathcal{C}$ is **strong** if it has a natural transformation

$$st_{A,B} : A \otimes TB \longrightarrow T(A \otimes B) \qquad \text{"strength map"}$$

such that

$$\begin{tikzcd} I \otimes TA \arrow[r, "st_{I,A}"] \arrow[dr, "\lambda_{TA}"'] & T(I \otimes A) \arrow[d, "T\lambda_A"] \\ & TA \end{tikzcd} \qquad \begin{tikzcd} A \otimes B \arrow[r, "id_A \otimes \eta_B"] \arrow[dr, "\eta_{A \otimes B}"'] & A \otimes TB \arrow[d, "st_{A,B}"] \\ & T(A \otimes B) \end{tikzcd}$$

$$\begin{tikzcd} (A \otimes B) \otimes TC \arrow[rr, "st_{A \otimes B, C}"] \arrow[d, "\alpha_{A,B,TC}"] & & T((A \otimes B) \otimes C) \arrow[d, "T\alpha_{A,B,C}"] \\ A \otimes (B \otimes TC) \arrow[r, "id_A \otimes st_{B,C}"'] & A \otimes T(B \otimes C) \arrow[r, "st_{A, B \otimes C}"'] & T(A \otimes (B \otimes C)) \end{tikzcd}$$

$$\begin{tikzcd} A \otimes T^2 B \arrow[r, "st_{A,TB}"] \arrow[d, "id_A \otimes \mu_B"] & T(A \otimes TB) \arrow[r, "Tst_{A,B}"] & T^2(A \otimes B) \arrow[d, "\mu_{A \otimes B}"] \\ A \otimes TB \arrow[rr, "st_{A,B}"'] & & T(A \otimes B) \end{tikzcd}$$

Def: T is commutative if it is strong and

$$
\begin{tikzcd}
 & T(T(A)\otimes B) \arrow[r, "T(st'_{A,B})"] & T^2(A\otimes B) \arrow[dr, "\mu_{A\otimes B}"] & \\
TA\otimes TB \arrow[ur, "st_{TA,B}"] \arrow[dr, "st'_{A,TB}"'] \arrow[rrr, dashed, "dst"] & & & T(A\otimes B) \\
 & T(A\otimes TB) \arrow[r, "T(st_{A,B})"'] & T^2(A\otimes B) \arrow[ur, "\mu_{A\otimes B}"'] &
\end{tikzcd}
$$

where

$$
\begin{tikzcd}
TA\otimes B \arrow[r, dashed, "st'_{A,B}"] \arrow[d, "\gamma_{TA,B}"'] & T(A\otimes B) \\
B\otimes TA \arrow[r, "st_{B,A}"'] & T(B\otimes A) \arrow[u, "T\gamma_{B,A}"']
\end{tikzcd}
$$

Lem: $Kl(T)$ is symmetric monoidal $\Longleftrightarrow$ T is commutative

Ex:

Maybe is strong: $A \times (B+1) \longrightarrow (A \times B) + 1$

Writer is strong: $A \times (B \times O) \longrightarrow (A \times B) \times O$

Writer is commutative $\Longleftrightarrow$ $O \times O \to O$ is commutative

any monad on Set is strong:
$$A \times TB \longrightarrow T(A \times B)$$
$$(a, t) \longmapsto T(\lambda b.\ (a,b))(t)$$

Def: a distributive law of monad $S : \mathcal{C} \to \mathcal{C}$ over a monad $T : \mathcal{C} \to \mathcal{C}$ is a natural transformation $\delta_A : TSA \to STA$ such that

$$\begin{tikzcd}
TSSA \arrow[r, "\delta_{SA}"] \arrow[d, "T\mu^S_A"'] & STSA \arrow[r, "S\delta_A"] & SSTA \arrow[d, "\mu^S_{TA}"] \\
TSA \arrow[rr, "\delta_A"'] & & STA \\
TTSA \arrow[u, "\mu^T_{SA}"] \arrow[r, "T\delta_A"'] & TSTA \arrow[r, "\delta_{TA}"'] & STTA \arrow[u, "S\mu^T_A"']
\end{tikzcd}$$

$$\begin{tikzcd}
& TA \arrow[dl, "T\eta^S_A"'] \arrow[dr, "\eta^S_{TA}"] & \\
TSA \arrow[rr, "\delta_A"'] & & STA
\end{tikzcd}
\qquad
\begin{tikzcd}
& SA \arrow[dl, "\eta^T_{SA}"'] \arrow[dr, "S\eta^T_A"] & \\
TSA \arrow[rr, "\delta_A"'] & & STA
\end{tikzcd}$$

Thm: if δ is distributive law, then $ST : \mathcal{C} \to \mathcal{C}$ is a monad

Ex: $S = T =$ nonempty List monad

$$\begin{tikzcd}
 & {[[a,b,e]]} \\
{[[a],[b,c,d],[e,f]]} \arrow[ur, "\delta_1"] \arrow[r, "\delta_2"] \arrow[dr, "\delta_3"'] & {[[a,d,f]]} \\
 & {[[a,b],[c],[d,e],[f]]}
\end{tikzcd}$$

$[[a,b,e]]$: take first elements

$[[a,d,f]]$: take last elements

$[[a,b],[c],[d,e],[f]]$: swap ',' and '],['

Ex: Powerset does <u>not</u> distributive over Distribution

so nondeterminism and probability hard to combine