Type Theory and Implicit Computational Complexity

Recap

i) Linear λ-calculus

ii) Add special data types get PTIME

iii) Realisability used to prove PTIME soundness

Today Combine linear and dependent types

$$x : \mathrm{Nat},\ y : \mathrm{Fin}\ x \vdash y : \mathrm{Fin}\ x$$

- In TT : variables have mixed "computational" and "logical" uses
- In LL : presence of a variable (or not) is important.

$$x : \mathrm{Nat} \mid y : \mathrm{Fin}\ x \vdash y : \mathrm{Fin}\ x$$

$\Gamma \vdash M : A$ "intuitionistic" or "unrestricted"

$\Gamma \mid \Delta \vdash L : X$ "linear"

$$\rho + \pi = 0 \Rightarrow \rho = 0 \wedge \pi = 0$$

$$x_1 \overset{\rho_1}{:} S_1, \ldots, x_n \overset{\rho_n}{:} S_n \vdash M : T \qquad \rho \cdot \pi = 0 \Rightarrow \rho = 0 \vee \pi = 0$$

$$\Gamma \vdash M \overset{0}{:} T \qquad\qquad \Gamma \vdash M \overset{1}{:} T$$

$$\frac{\Gamma \vdash M \overset{1}{:} T}{0\Gamma \vdash M \overset{0}{:} T}\ 0\text{-ing} \qquad (\text{admissible})$$

This system is called "Quantitative Type Theory".

QTT + ConsFree

$$\frac{0\Gamma \vdash}{0\Gamma \vdash \mathsf{ze} \overset{0}{:} \mathrm{Nat}^{\bullet}} \qquad \frac{0\Gamma \vdash N \overset{0}{:} \mathrm{Nat}^{\bullet}}{0\Gamma \vdash \mathsf{su}\ N \overset{0}{:} \mathrm{Nat}^{\bullet}}$$

$$\frac{\begin{array}{l} 0\Gamma, x \overset{0}{:} \mathrm{Nat}^{\bullet} \vdash P \ \mathrm{type} \\ 0\Gamma \vdash M_z \overset{\sigma}{:} P[{}^{\mathsf{ze}}/_{x}] \\ 0\Gamma, n \overset{0}{:} \mathrm{Nat}^{\bullet}, r \overset{\sigma}{:} P[{}^{n}/_{x}] \vdash M_s \overset{\sigma}{:} P[{}^{\mathsf{su}\ n}/_{x}] \\ \Gamma \vdash N \overset{\sigma}{:} \mathrm{Nat}^{\bullet} \end{array}}{\Gamma \vdash \mathrm{iter}(x.\,P, M_z, n\,r.\,M_s, N) \overset{\sigma}{:} P[{}^{N}/_{x}]}$$

QTT + LFPL

$$\frac{\bigcirc\Gamma \vdash}{\bigcirc\Gamma \vdash \Diamond\ \mathsf{type}} \qquad \frac{}{\bigcirc\Gamma \vdash * \overset{\circ}{:} \Diamond} \qquad \frac{\bigcirc\Gamma \vdash M \overset{\circ}{:} \Diamond}{\bigcirc\Gamma \vdash M \equiv * \overset{\circ}{:} \Diamond}$$

$$\frac{\Gamma \vdash D \overset{\sigma}{:} \Diamond}{\Gamma \vdash \mathsf{ze}@D : \mathsf{Nat}^{\bullet}} \qquad \frac{\Gamma_1 \vdash N \overset{\sigma}{:} \mathsf{Nat}^{\bullet} \quad \Gamma_2 \vdash D \overset{\sigma}{:} \Diamond}{\Gamma_1 + \Gamma_2 \vdash \mathsf{su}\, N@D \overset{\sigma}{:} \mathsf{Nat}^{\bullet}}$$

$$\frac{\begin{array}{c} \bigcirc\Gamma, x \overset{\circ}{:} \mathsf{Nat}^{\bullet} \vdash P\ \mathsf{type} \\ \bigcirc\Gamma, d \overset{\sigma}{:} \Diamond \vdash M_z \overset{\sigma}{:} P[\mathsf{ze}@*/x] \\ \bigcirc\Gamma, d \overset{\sigma}{:} \Diamond, n \overset{\circ}{:} \mathsf{Nat}^{\bullet}, r \overset{\sigma}{:} P[n/x] \vdash M_s \overset{\sigma}{:} P[\mathsf{su}\, n@*/x] \\ \Gamma \vdash N \overset{\sigma}{:} \mathsf{Nat}^{\bullet} \end{array}}{\Gamma \vdash \mathsf{iter}(x.P, M_z, d\, n\, r.\, M_s, N) \overset{\sigma}{:} P[N/x]}$$

$$\frac{\bigcirc\Gamma \vdash M :^{1} A}{\bigcirc\Gamma \vdash R(M) :^{0} R(A)} \qquad \frac{\Gamma \vdash M :^{0} R(A)}{\Gamma \vdash R^{-1}(M) :^{0'} A}$$

A problem : $(A : \mathrm{Set}\ ,\ P : A \to \mathrm{Set})$

$$\mathrm{PTIME}(A, P) = \sum f : R(A \multimap \mathrm{Bool}).$$
$$\Pi a : A.\ (R^{-1}(f)_a = \mathrm{true} \Leftrightarrow P\,a)$$

— Define poly-time reduction between problems

— Define further classes.