Type Theory and Implicit Computational Complexity

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LECTURE 3

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Recap

Realisability models.

1. Assume $\underbrace{\mathcal{E}, \mathcal{V} \text{ sets of programs and values}}_{}$ "realisers"
   with $\mathrm{eval} : \mathcal{E} \times \mathcal{V} \times \mathcal{V} \longrightarrow \mathrm{Prop}$
2. Interpret types as pairs $(|A|, \Vdash_A \subseteq \mathcal{V} \times |A|)$
3. Interpret terms as $(|A|, \Vdash_A) \xrightarrow{f} (|B|, \Vdash_B)$ comprising
   $|f| : |A| \longrightarrow |B|$
   s.t. $\exists e \in \mathcal{E}$.
   $\forall a, v .\ v \Vdash_A a \Longrightarrow \exists v'\ \mathrm{eval}(e, v, v') \wedge v' \Vdash |f|(a)$.

Goal: Adapt this set up to prove time bounds.

We need:

- Accounting for
  a) sizes
  b) "potential"

- Notion of evaluation that is costed

- Put them together

Resource monoids $\mathcal{M}$

- A carrier set $|\mathcal{M}| \ni 0$, $+ : |\mathcal{M}| \times |\mathcal{M}| \longrightarrow |\mathcal{M}|$
- $\mathcal{Q}_\mathcal{M} : |\mathcal{M}| \times |\mathcal{M}| \longrightarrow \mathbb{N} \cup \{-\infty\}$

$$\mathcal{Q}_\mathcal{M}(\alpha, \alpha) \geqslant 0$$

$$\mathcal{Q}_\mathcal{M}(\alpha, \beta) + \mathcal{Q}_\mathcal{M}(\beta, \gamma) \leqslant \mathcal{Q}_\mathcal{M}(\alpha, \gamma)$$

$$\mathcal{Q}_\mathcal{M}(\alpha, \beta) \leqslant \mathcal{Q}_\mathcal{M}(\alpha + \gamma, \beta + \gamma)$$

$$\mathcal{Q}_\mathcal{M}((\alpha + \beta) + \gamma, \alpha + (\beta + \gamma)) = 0$$

Basic example:

$|\mathcal{M}| = \mathbb{N}$

$\mathcal{Q}_\mathbb{N}(\alpha, \beta) = \alpha - \beta$

$$\mathrm{acct} : \mathbb{N} \longrightarrow |\mathcal{M}| \qquad \text{s.t. } \mathcal{Q}_\mathcal{M}(\mathrm{acct}(n), 0) = n$$

- A sub resource monoid $\mathcal{M}_0 \leq \mathcal{M}$ $\quad |\mathcal{M}_0| \subseteq |\mathcal{M}|$

Assume $\mathcal{E}, \mathcal{V}$ with eval : $\mathcal{E} \times \mathcal{V} \times \mathcal{V} \to \mathbb{N} \cup \{\infty\}$

- Types are interpreted as $(|A|, \vDash_A \subseteq \mathcal{V} \times |M| \times |A|)$

- Terms are interpreted as $(|A|, \vDash_A) \xrightarrow{f} (|B|, \vDash_B)$

$$|f| : |A| \longrightarrow |B|$$

s.t. $\exists e \in \mathcal{E}.\ \exists \gamma \in |M_0|$

$\forall v, \alpha, a.\ v, \alpha \vDash_A a \Rightarrow$

$\exists v', \beta.\ \text{eval}(e, v, v') \leq \mathcal{Q}_M(\alpha + \gamma, \beta)$

and $v', \beta \vDash_B |f|(a)$.

$$A \otimes B = \Big(|A| \times |B|,$$
$$\Big\{ v, \alpha \models (a, b) \;\Big|\; \exists v_1, v_2, \beta_1, \beta_2 .\; v = (v_1, v_2),\; D_m(\alpha, \beta_1 + \beta_2) \geq 0,\; v_1, \beta_1 \models a,\; v_2, \beta_2 \models b \Big\} \Big)$$

$$A \multimap B = \Big(|A| \longrightarrow |B|,$$
$$\Big\{ v, \gamma \models f \;\Big|\; \forall v', \alpha, a .\; v', \alpha \models_A a \Longrightarrow \exists v'', \beta .\; \mathrm{eval}(\mathrm{app}(v), v', v'') \leq D(\alpha + \gamma, \beta) \text{ and } v'', \beta \models f(a) \Big\} \Big)$$

in m!

$$\frac{\Gamma \vdash M : A \qquad \Delta \vdash N : B}{\Gamma, \Delta \vdash (M, N) : A \otimes B}$$

$$|M| : |\Gamma| \longrightarrow |A| \qquad\qquad |N| : |\Delta| \longrightarrow |B|$$

e_M, γ_M $\qquad\qquad$ e_N, γ_N

program $\quad$ resource

$$|(M,N)| = \lambda(g,d).(|M| g, |N| d)$$

$$\begin{aligned} e_{(M,N)} = \; & \text{let } (x_1, x_2) = x \\ & \text{let } y_1 = e_M\, x_1 \\ & \text{let } y_2 = e_N\, x_2 \\ & (y, y_2) \end{aligned}$$

$$\begin{aligned} \gamma_{(M,N)} = \; & \mathrm{acct}(1) + \\ & \gamma_M + \\ & \mathrm{acct}(1) + \\ & \gamma_N + \\ & \mathrm{acct}(1) + \\ & \mathrm{acct}(1) \end{aligned}$$

Polynomial Time Resource Monoids.

(n, p) $\quad n \in \mathbb{N}$

$\quad\quad\quad\quad p \in \mathbb{N}[x]$

ConsFree

MaxPoly: $(n, p) + (m, q) = (n \sqcup m, p + q)$

LFPL

SumPoly: $(n, p) + (m, q) = (n + m, p + q)$

$$\mathcal{D}((a, p), (b, q)) = \begin{cases} p(a) - q(a) & \text{if } a \geq b \text{ and } \forall k \geq a \; p(k) \geq q(k) \\ -\infty & \text{otherwise} \end{cases}$$

$\mathrm{acct}(a) = (0, a)$

$M_0 = \{(0, p) \mid p \in \mathbb{N}[x]\}$.