Type Theory and Implicit Computational Complexity

Lecture 2

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$\mathrm{Nat}^{\bullet}$ — iterable but not constructible

Nat° — constructible but not iterable

$$\frac{\vdash M_z : A \qquad x : A \vdash M_s : A \qquad \Gamma \vdash N : \mathrm{Nat}^{\bullet}}{\Gamma \vdash \mathrm{iter}(M_z,\ x.\ M_s,\ N) : A}$$

$$\frac{\Gamma \vdash N : \mathrm{Nat}^{\bullet}}{\Gamma \vdash \mathrm{depNat}\ N : \mathrm{Nat}^{\bullet} \otimes \mathrm{Nat}^{\bullet}}$$

$$\frac{}{\vdash \mathrm{ze} : \mathrm{Nat}^{\circ}} \qquad \frac{\Gamma \vdash N : \mathrm{Nat}^{\circ}}{\Gamma \vdash \mathrm{su}\ N : \mathrm{Nat}^{\circ}}$$

$$\frac{\Gamma \vdash M_z : A \qquad \Gamma, x : \mathrm{Nat}^{\circ} \vdash M_s : A \qquad \Delta \vdash N : \mathrm{Nat}^{\circ}}{\Gamma, \Delta \vdash \mathrm{case}(M_z,\ x.\ M_s,\ N) : A}$$

$f : \mathrm{TAPE} \multimap \mathrm{TAPE}$ $p(n)$

$\mathrm{List}(\mathrm{Bool}) \otimes \mathrm{Bool} \otimes \mathrm{List}(\mathrm{Bool})$

$I_0 : \mathrm{Nat}^{\bullet} \multimap \mathrm{TAPE} \multimap \mathrm{TAPE}$

$I_0 = \mathrm{iter}(\lambda t.\, t,\ r.\, \lambda t.\, f(r\, t))$

$I_{d+1} : \mathrm{Nat}^{\bullet} \multimap \mathrm{TAPE} \multimap \mathrm{TAPE}$

$I_{d+1} = \lambda n.\, \lambda t.$
$\quad \mathrm{let}\ (n, n') = \mathrm{dupNat}\ n\ \mathrm{in}$
$\quad \mathrm{iter}(\lambda(n,t).\, t,\ r.\, \lambda(n,t).\ \mathrm{let}\ (n,n') = \mathrm{dupNat}\ n\ \mathrm{in}$
$\quad\quad r\, n\, (I_d(n')(t)))$

$n\ n'\ t$

$I_d(n) : \mathrm{TAPE} \multimap \mathrm{TAPE}$ iterates f n^{d+1}

$f : \mathrm{Nat}^{\bullet} \multimap A$

$\diamond$ type ~~dpNaF~~

$$\frac{\Gamma \vdash d : \diamond}{\Gamma \vdash ze@d : Nat^\bullet} \qquad \frac{\Gamma \vdash N : Nat^\bullet \qquad \Delta \vdash d : \diamond}{\Gamma, \Delta \vdash su\,N@d : Nat^\bullet}$$

$$\frac{d:\diamond \vdash M_z : A \qquad x:A, d:\diamond \vdash M_s : A \qquad \Gamma \vdash N : Nat^\bullet}{\Gamma \vdash iter(M_z, x\,d.\,M_s, N) : A}$$

- Pay $n+1$ $\diamond$s for construction, but n $\diamond$s back! linear time
- if $\diamond$ = memory location, then this can be implemented via in-place update.
- This system is called LFPL.

Soundness

$$\vdash M : Nat^{\bullet} \multimap Bool$$

then there exists a program e_M and polynomial p s.t.

$\forall n.$ $eval(e_M, n) \rightsquigarrow b$, R steps

s.t. $R \leq p(n)$.

Realisability

1. $[\![Nat^{\bullet}]\!] = \mathbb{N}$

   $[\![Nat^{\circ}]\!] = \mathbb{N}$

   $[\![A \multimap B]\!] = [\![A]\!] \to [\![B]\!]$

2. Assume some set of programs $\mathcal{E}$ and values $\mathcal{V}$

   $eval : \mathcal{E} \times \mathcal{V} \times \mathcal{V} \to Prop$

3. Type is $A = (|A| : \mathrm{Set}, \vDash_A \subseteq \mathcal{V} \times |A|)$

Term $A \vdash B$ $\quad f : |A| \longrightarrow |B|$ s.t. $\exists e \in \mathcal{E}.$

$$\forall a, v.\ v \vDash_A a \Rightarrow \exists v'.\ \mathrm{eval}(e, v, v') \wedge v' \vDash_B f(a)$$

<u>Examples</u> 1) Let $\mathcal{V} = \mathbb{N}$, $\mathcal{E} = \{*\}$ $\qquad \mathrm{eval}(*, n, n') \Leftrightarrow n' \leq n$

$$[\![\mathrm{Nat}^{\bullet}]\!] = (\mathbb{N}, \{ n \vDash n' \mid n \geq n' \})$$

$$[\![\mathrm{Nat}^{\circ}]\!] = (\mathbb{N}, \{ n \vDash n' \mid n \geq 0 \})$$

$$[\![A \otimes B]\!] = (|A| \times |B|, \{ n \vDash (a, b) \mid \exists n_1, n_2.\ n_1 + n_2 \leq n,\ n_1 \vDash_A a,\ n_2 \vDash_B b \})$$

$$[\![A \times B]\!] = (|A| \times |B|, \{ n \vDash (a, b) \mid n \vDash a \text{ and } n \vDash b \})$$

Example 2 $\quad \mathcal{E} = \mathcal{V} \ni B, I \qquad (\cdot) : \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{E}$

$$\left.\begin{aligned} B \cdot x \cdot y \cdot z &= x \cdot (y \cdot z) \\ I \cdot x &= x \end{aligned}\right\} \text{ enough for a category}$$

$$f : X \vdash Y \qquad\qquad g : Y \vdash Z \qquad\qquad g \circ f$$

$$\alpha \qquad\qquad \beta \qquad\qquad B \cdot \beta \cdot \alpha$$

$$\mathrm{id} : X \vdash X$$

$$\| $$

$$I$$

- If we add $C \cdot x \cdot y \cdot z = x \cdot z \cdot y$ then we get $\otimes$-products and $\multimap$.